Review

Processing Information in Quantum Decision Theory

Vyacheslav I. Yukalov [1,2] **and Didier Sornette** [1,3,*]

[1] Department of Management, Technology and Economics, ETH Zürich, Zürich CH-8032, Switzerland;
E-Mail: syukalov@ethz.ch

[2] Bogolubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna 141980,
Russia; E-Mail: yukalov@theor.jinr.ru

[3] Swiss Finance Institute, c/o University of Geneva, 40 blvd. Du Pont d'Arve, CH 1211 Geneva 4,
Switzerland

[*] Author to whom correspondence should be addressed; E-Mail: dsornette@ethz.ch;
Tel.: +41-04463-28917; Fax: +41-04463-21914.

Abstract: A survey is given summarizing the state of the art of describing information processing in Quantum Decision Theory, which has been recently advanced as a novel variant of decision making, based on the mathematical theory of separable Hilbert spaces. This mathematical structure captures the effect of superposition of composite prospects, including many incorporated intended actions. The theory characterizes entangled decision making, non-commutativity of subsequent decisions, and intention interference. The self-consistent procedure of decision making, in the frame of the quantum decision theory, takes into account both the available objective information as well as subjective contextual effects. This quantum approach avoids any paradox typical of classical decision theory. Conditional maximization of entropy, equivalent to the minimization of an information functional, makes it possible to connect the quantum and classical decision theories, showing that the latter is the limit of the former under vanishing interference terms.

Classification: PACS 03.65.Aa, 03.65.Ta, 03.67.Ac, 03.67.Bg, 03.67.Hk

1. Basic Ideas and Historical Retrospective

This section serves as an introduction to the problem, describing the historical retrospective, related studies, and basic ideas of the quantum approach to decision making. First of all, in order that the reader would not be lost in details, we need to stress the main goal of the approach.

Principal Goal: *The principal goal of the quantum approach to decision making is to develop a unified theory that, from one side, could formalize the process of taking decisions by human decision makers in terms of quantum language and, from another side, would suggest a scheme of thinking quantum systems that could be employed for creating artificial intelligence.*

Generally, decision theory is concerned with identifying what are the optimal decisions and how to reach them. Traditionally, it is a part of discrete mathematics. Most of decision theory is normative and prescriptive, and assumes that people are fully-informed and rational. These assumptions have been questioned early on with the evidence provided by the Allais paradox [1] and many other behavioral paradoxes [2], showing that humans often seem to deviate from the prescription of rational decision theory due to cognitive and emotion biases. The theories of bounded rationality [3], of behavioral economics and of behavioral finance have attempted to account for these deviations. As reviewed by Machina [4], alternative models of preferences over objectively or subjectively uncertain prospects have attempted to accommodate these systematic departures from the expected utility model, while retaining as much of its analytical power as possible. In particular, non-additive nonlinear probability models have been developed to account for the deviations from objective to subjective probabilities observed in human agents [5–10]. However, many paradoxes remain unexplained or are sometimes rationalized on an ad hoc basis, which does not provide much predictive power.

Another approach to decision theory can be proposed, being part of the mathematical theory of Hilbert spaces [11] and employing the mathematical techniques that are used in quantum theory. (see, e.g., the special issue [12] and references therein). However, no self-consistent quantum theory of decision making has been developed, which would have predictive power.

Recently, we introduced a general framework, called the *Quantum Decision Theory* (QDT), in which decisions involve composite intended actions which, as we explain below, provides a unifying explanation of many paradoxes of classical decision theory in a quantitative predictive manner [13]. Such an approach can be thought of as the mathematically simplest and most natural extension of objective probabilities into nonlinear subjective probabilities. The proposed formalism allows one to explain *quantitatively* different anomalous phenomena, e.g., the disjunction and conjunction effects. The disjunction effect is the failure of humans to obey the sure-thing principle of classical probability theory. The conjunction effect is a logical fallacy that occurs when people assume that specific conditions are more probable than a single general one. The QDT unearths a deep relationship between the conjunction fallacy and the disjunction effect, the former being sufficient for the latter to exist.

QDT uses the same underlying mathematical structure as the one developed to establish a rigorous formulation of quantum mechanics [14]. Based on the mathematical theory of separable Hilbert spaces on the continuous field of complex numbers, quantum mechanics showed how to reconcile and combine the continuous wave description with the fact that waves are organized in discrete energy packets, called quanta, which behave in a manner similar to particles. Analogously, in the QDT framework,

the qualifier quantum emphasizes the fact that a decision is a discrete selection from a large set of entangled options. The key idea of QDT is to provide the simplest generalization of the classical probability theory underlying decision theory, so as to account for the complex dynamics of the many nonlocal hidden variables that may be involved in the cognitive and decision making processes of the brain. The mathematical theory of complex separable Hilbert spaces provides the simplest direct way to avoid dealing with the unknown hidden variables, and at the same time reflecting the complexity of nature [15]. In decision making, unknown states of nature, emotions, and subconscious processes play the role of hidden variables.

Before presenting the QDT approach, it is useful to briefly summarize previous studies of decision making and of the associated cognitive processes of the brain which, superficially, could be considered as related to the QDT approach. This exposition will allow us to underline the originality and uniqueness of the approach. We do not touch here purely physiological aspects of the problem, which are studied in medicine and physiological cognitive sciences. Concerning the functional aspects of the brain, we focus our efforts towards its formal mathematical modeling.

One class of approaches is based on the theory of neural networks and of dynamical systems (see, e.g., [16–19]). These bottom-up approaches suffer from the obvious difficulties of modeling the emergence of upper mental faculties from a microscopic constructive neuron-based description.

Two main classes of theories invoke the qualifier "quantum". In the first class, one finds investigations which attempt to represent the brain as a quantum or quantum-like object [20–22], for which several mechanisms have been suggested [23–29]. The existence of genuine quantum effects and the operation of any of these mechanisms in the brain remain however controversial and have been criticized by Tegmark as being unrealistic [30]. Another approach in this first class appeals to the mind-matter duality, treating mind and matter as complementary aspects and considering consciousness as a separate fundamental entity [31–34]. This allows one, without insisting on the quantum nature of the brain processes, if any, to ascribe quantum properties solely to the consciousness itself, as has been advocated by Stapp [35, 36]. Actually, the basic idea that mental processes are similar to quantum-mechanical phenomena goes back to the founder of the old quantum mechanics, Niels Bohr. One of the first publications on this analogy is his paper [37]. Later on, he returned many times to the similarity between quantum mechanics and the function of the brain, for instance in [38–40]. This analogy proposes that mental processes could be modeled by a quantum-mechanical wave function, whose evolution would be characterized by a dynamical equation, like the Schrödinger equation.

The second class of theories does not necessarily assume quantum properties of the brain or that consciousness is a separate entity with quantum characteristics. Rather, these approaches use quantum techniques, as a convenient language to generalize classical probability theory. An example is provided by the quantum games [41–50]. With the development of quantum game theory, it has been shown that many quantum games can be reformulated as classical games by allowing for a more complex game structure [51–54]. But, in the majority of cases, it is more efficient to play quantum game versions, as less information needs to be exchanged. Another example is the Shor algorithm [55], which is purely quantum-mechanical but is solving the classical factoring problem. This shows that there is no contradiction in using quantum techniques to describe classical problems. Here "classical"

is contrasted with "quantum", in the sense consecrated by decades of discussions on the interpretation of quantum mechanics. In fact, some people go as far as stating that quantum mechanics is nothing but an effective theory describing very complicated classical systems [56–58]. Interpretations of this type have been made, e.g., by de Broglie and Bohm. An extensive literature in this direction can be found in de Broglie [59] and Bohm [60]. In any case, whether we deal really with a genuinely quantum system or with an extremely complex classical system, the language of quantum theory can be a convenient effective tool for describing such systems [15]. In the case of decision making performed by real people, the subconscious activity and the underlying emotions, which are difficult to quantify, play the role of the hidden variables appearing in quantum theory.

The QDT belongs to this second class of theories, *i.e.*, we use the construction of complex separable Hilbert spaces as a *mathematical language* that is convenient for characterizing the complicated processes in the mind, which are associated with decision making. This approach encompasses in a natural way several delicate features of decision making, such as its probabilistic nature, the existence of entangled decisions, the possible non-commutativity of decisions, and the interference between several different decisions. These terms and associated concepts are made operationally clear in the sequel.

As a bonus, the QDT provides natural algorithms which could be used in the future for quantum information processing, the operation of quantum computers, and in creating artificial intelligence.

The classical approaches to decision making are based on utility theory [61, 62]. Decision making in the presence of uncertainty about the states of nature is formalized in the statistical decision theory [63–73]. Some problems, occurring in the interpretation of the classical utility theory and its application to real human decision processes have been discussed in numerous literature (e.g., [4, 68, 74, 75]).

Quantum approach to decision making, suggested in Reference [13], is principally different from the classical utility theory. In this approach, the action probability is defined as is done in quantum mechanics, using the mathematical theory of complex separable Hilbert spaces. This proposition can be justified by invoking the following analogy. The probabilistic features of quantum theory can be interpreted as being due to the existence of the so-called nonlocal hidden variables. The dynamical laws of these nonlocal hidden variables could be not merely extremely cumbersome, but even not known at all, similarly to the unspecified states of nature in decision theory. The formalism of quantum theory is then formulated in such a way as to avoid dealing with unknown hidden variables, but at the same time, to reflect the complexity of nature [15]. In decision making, the role of hidden variables is played by unknown states of nature, by emotions, and by subconscious processes, for which quantitative measures are not readily available.

In the following sections, we develop the detailed description of the suggested program, explicitly constructing the action probability in quantum-mechanical terms. The probability of an action is intrinsically subjective, as it must characterize intended actions by human beings. For brevity, an intended action can be called an *intention* or just an *action*. And, in compliance with the terminology used in the theories of decision-making, a composite set of intended actions, consisting of several sub-actions, is called a *prospect*. An important feature of the quantum approach is that, in general, it deals not with separate intended actions, but with composite prospects, including many incorporated intentions. Only then it becomes possible, within the frame of one general theory, to describe a variety

of interesting unusual phenomena that have been reported to characterize the decision making properties of real human beings.

The pivotal point of the approach, formalized in QDT, is that mathematically it is based on the von Neumann theory of quantum measurements [14]. The formal relation of the von Neumann measurement theory to quantum information processing has been considered by [76]. QDT generalizes the quantum measurement theory to be applicable not merely to simple actions, but also to composite prospects, which is of paramount importance for the appearance of decision interference. The principal difference of QDT from the measurement theory is the existence of a specific *strategic state* characterizing each particular decision maker.

A brief account of the axiomatics of QDT has been published in the recent letter [13]. The principal scheme of functioning of a thinking quantum system, imitating the process of decision making, has been advanced [77]. The applicability of the suggested quantum approach for analyzing the phenomena of dynamic inconsistency has been illustrated [78]. The aim of the present survey is to provide a detailed explanation of the theory and to demonstrate that it can be successfully applied to real-life problems of decision making. We also show that the method of conditional entropy maximization, which is equivalent to the minimization of an information functional, yields an explicit relation between the quantum decision theory and the classical decision theory based on the standard notion of expected utility.

2. Mathematical Foundation of Quantum Decision Theory

In order to formulate in precise mathematical terms the scheme of information processing and decision making in quantum decision theory, it is necessary to introduce several definitions. To better understand these definitions, we shall give some very simple examples, although much more complicated cases can be invented. The entity concerned with the decision making task can be a single human, a group of humans, a society, a computer, or any other system that is able or enables to make decisions. Throughout the paper, for the operations with intended actions, we shall use the notations that are accepted in the literature on decision theory [62–73] and for the physical states, we shall employ the Dirac notations widely used in quantum theory [79].

Definition 1. Action ring

The process of taking decisions implies that one is deliberating between several admissible actions with different outcomes, in order to decide which of the intended actions to choose. Therefore, the first element arising in decision theory is an *intended action A*.

An intended action which, for brevity, can be called an intention or an action, is a particular thought about doing something. Examples of intentions could be as follows: "I would like to marry" or "I would like to be rich" or "I would like to establish a firm". There can be a variety of intentions, which we assume to be enumerated by an index $i = 1, 2, 3, \ldots, N$, where the total number N of actions can be finite or infinite.

The whole family of all these actions forms the *action set*

$$\mathcal{A} \equiv \{A_i : \ i = 1, 2, \ldots, N\} . \tag{1}$$

The elements of this set are assumed to be endowed with two binary operations, addition and multiplication, so that, if A and B pertain to $\mathcal{A}$, then AB and $A + B$ also pertain to $\mathcal{A}$. The addition is associative, such that $A + (B + C) = (A + B) + C$, and reversible, in the sense that $A + B = C$ implies $A = C - B$. The multiplication is distributive, $A(B + C) = AB + BC$. The multiplication is not necessarily commutative, so that, generally, AB is not the same as BA.

Among the elements of the action set (1), there is an identity action 1, for which $A1 = 1A = A$. The identity action 1 is not to "do nothing", since inaction is actually an action. This is well recognized for instance in the field of risk management. Consider for instance the famous quotes: "The man who achieves makes many mistakes, but he never makes the biggest mistake of all—doing nothing" (Benjamin Franklin), or "Life is inherently risky. There is only one big risk you should avoid at all costs, and that is the risk of doing nothing" (Denis Waitley). This also resonates with the standard recommendations in risk management: "If you do not actively attack risks, they will attack you" or "Risk prevention is cheaper than reconstruction." Thus, "not acting" is not the identity action. We interpret the identity action 1 as the action of keeping running the present action an individual is involved in. For instance, if action A is "to marry someone", the action $1A$ is to marry someone and to confirm this action. The action $A1$ can be interpreted as first "being open to decide an action" and then to "decide to marry someone".

And there exists an impossible action 0, for which $A0 = 0A = 0$. Two actions are called disjoint, when their joint action is impossible, giving $AB = BA = 0$. The action set (1), with the described structure, is termed the *action ring*.

We recall that, in mathematics, a ring is an algebraic structure consisting of a set together with two binary operations (usually called addition and multiplication), where each operation combines two elements to form a third element (closure property). This closure property here embodies the fact that choosing between alternative actions or combining several actions still correspond to actions.

In the algebra of the elements of the action ring, the meaning of the operations of addition and multiplication is the same as is routinely used in the literature [62–73]. The sum $A + B$ means that either the action A or action B is intended to be realized. And the product of actions AB implies that both these actions are to be accomplished together. Instead of writing the sum $A_1 + A_2 + \cdots$, it is often convenient to use the shorter summation symbol $\bigcup_i A_i \equiv A_1 + A_2 + \cdots$, which is also the standard abbreviation. Similarly, for a long product $A_1 A_2 \cdots$, it is convenient to use the shorter notation $\bigcap_i A_i \equiv A_1 A_2 \cdots$. The use of these standard notations can lead to no confusion.

Definition 2. Action modes

An action is simple, when it cannot be decomposed into the sum of other actions. An action is composite, when it can be represented as a sum of several other actions. If an action is represented as a sum

$$A_i = \bigcup_{\mu=1}^{M_i} A_{i\mu} \; , \tag{2}$$

whose terms are mutually incompatible, then these terms are named the *action modes*. Here, M_i denotes the number of modes in action A_i. The modes correspond to different possible ways of realizing an action. According to the meaning emphasized above, the summation symbol in Equation (2) implies that one of the actions is intended to be realized.

Action representations, or action modes, are concrete implementations of an intended action. For instance, the intention "to marry" can have as representations the following variants: "to marry A" or "to marry B", and so on. The intention "to be rich" can have as representations "to be rich by working hard" or "to be rich by becoming a bandit". The intention "to establish a firm" can have as representations "to establish a firm producing cars" or "to establish a firm publishing books" and so on. We number all representations of an i-intended action by the index $\mu = 1, 2, 3, \ldots$. Note that intention representations may include not only positive intention variants "to do something" but also negative variants such as "not to do something". For example, Hamlet's hesitation "to be or not to be" is the intended action consisting of two representations, one positive and the other negative.

Definition 3. Elementary prospects

Generally, decision taking is not necessarily associated with a choice of just one action among several simple given actions, but it involves a choice between several complex actions. The simplest such complex action is defined as follows. Let the multi-index $n = \{\nu_1, \nu_2, ..., \nu_N\}$ be a set of indices enumerating several chosen modes, under the condition that each action is represented only by one of its modes. The *elementary prospect* is the conjunction

$$e_n \equiv \bigcap_{i=1}^{N} A_{i\nu_i} \; , \tag{3}$$

of the chosen modes, one for each of the actions from the action ring (1). The total set of all elementary prospects will be denoted as $\{e_n\}$.

Definition 4. Composite prospects

A prospect is composite, when it cannot be represented as an elementary prospect (3). Generally, a composite prospect is a conjunction

$$\pi_j = \bigcap_n A_{j_n} \tag{4}$$

of several composite actions of form (2), where each of the factors A_{j_n} pertains to the action ring (1). While expression (4) has the similar form as (3), the difference is that the actions A_{j_n} in (4) are composite while the actions $A_{i\nu_i}$ in (3) are elementary action modes.

A prospect is a set of several intended actions or several intention representations. In reality, a decision maker is always motivated by a variety of intentions, which are mutually interconnected. Even the realization of a single intention always involves taking into account many other related intended actions.

Definition 5. Prospect lattice

All possible prospects, among which one needs to make a choice, form a set

$$\mathcal{L} = \{\pi_j : \; j = 1, 2, \ldots, N_L\} \; . \tag{5}$$

The set is assumed to be equipped with the binary relations $>, <, =, \geq, \leq$, so that each two prospects π_i and π_j in $\mathcal{L}$ are related as either $\pi_i > \pi_j$, or $\pi_i = \pi_j$, or $\pi_i \geq \pi_j$, or $\pi_i < \pi_j$, or $\pi_i \leq \pi_j$. For a while, it is sufficient to assume that such an ordering exists. Then, the ordered set (5) is called a *lattice*. The explicit ordering procedure associated with decision making will be given below.

Definition 6. Mode space

To each action mode $A_{i\mu}$, there corresponds the *mode state* $|A_{i\mu}\rangle$, which is a complex function $\mathcal{A} \to \mathcal{C}$, and its Hermitian conjugate $\langle A_{i\mu}|$. Here we employ the Dirac notation [79]. We assume that a scalar product is defined, such that the mode states, pertaining to the same action, are orthonormalized:

$$\langle A_{i\mu}|A_{i\nu}\rangle = \delta_{\mu\nu} \; . \tag{6}$$

The *mode space* is the closed linear envelope

$$\mathcal{M}_i \equiv \mathrm{Span}\{|A_{i\mu}\rangle : \; \mu = 1, 2, \ldots, M_i\} \; , \tag{7}$$

spanning all mode states. By this definition, the mode space, corresponding to an i-action A_i, is a Hilbert space of dimensionality M_i. The elements of the mode space will be called the *intention states*.

Definition 7. Mind space

To each elementary prospect e_n, there corresponds the *basic state* $|e_n\rangle$, which is a complex function $\mathcal{A}^N \to \mathcal{C}$, and its Hermitian conjugate $\langle e_n|$. The structure of a basic state is

$$|e_n\rangle \equiv |A_{1\nu_1} A_{2\nu_2} \ldots A_{N\nu_N}\rangle = \bigotimes_{i=1}^{N} |A_{i\nu_i}\rangle \; . \tag{8}$$

The scalar product is assumed to be defined, such that the basic states are orthonormalized:

$$\langle e_m|e_n\rangle = \prod_{i=1}^{N} \delta_{\mu_i\nu_i} \equiv \delta_{mn} \; . \tag{9}$$

The *mind space* is the closed linear envelope

$$\mathcal{M} \equiv \mathrm{Span}\{|e_n\rangle\} = \bigotimes_{i=1}^{N} \mathcal{M}_i \; , \tag{10}$$

spanning all basic states (8). Hence, the mind space is a Hilbert space of dimensionality

$$\mathrm{dim}\mathcal{M} = \prod_{i=1}^{N} M_i \; .$$

The vectors of the mind space represent all possible actions and prospects considered by a decision maker.

The family of the basic states forms the *mind basis* $\{|e_n>\}$ in the mind space. Different states belonging to the mind basis are assumed to be disjoint, in the sense of being orthogonal. Since the modulus of each state has no special meaning, these states are also normalized to one. This is formalized as the orthonormality of the basis.

Definition 8. Prospect states

To each prospect π_j, there corresponds a state $|\pi_j\rangle \in \mathcal{M}$ that is a member of the mind space (10). Hence, the prospect state can be represented as an expansion over the basic states

$$|\pi_j\rangle = \sum_n a_{jn}|e_n\rangle \ . \tag{11}$$

The expansion coefficients in Equation (11) are assumed to be defined by the decision maker, so that $|a_{jn}|^2$ gives the weight of the state $|e_n>$ into the general prospect.

The prospects are enumerated with the index $j = 1, 2, \ldots$. The total set $\{|\pi_j>\}$ of all prospect states $|\pi_j>$, corresponding to all admissible prospects, forms a subset of the space of mind. The set $\{|\pi_j>\} \subset \mathcal{M}$ can be called the prospect-state set.

The prospect states are not required to be mutually orthogonal and normalized to one, so that the scalar product

$$\langle\pi_i|\pi_j\rangle = \sum_n a_{in}^* a_{jn}$$

is not necessarily a Kronecker delta. The normalization condition will be formulated for the prospect probabilities to be defined below.

The fact that different prospect states are not necessarily orthogonal assumes that the related prospects are not necessarily incompatible. The incompatibility is supposed only for the elementary prospects (3), whose states form the basis in the mind space (10) and are orthogonal to each other, according to Equation (9). But an arbitrarily defined composite prospect, generally, is not required to be orthogonal to all other considered prospects. In particular, this can be so, but, in general, we do not need this property.

The prospect states are not normalized to one, since, imposing such a condition would over define these states. The normalization condition will be imposed below on the prospect probabilities. Generally, imposing two normalization conditions could make them inconsistent with each other. So, we need just one normalization condition for the prospect probabilities, which is necessary for the correct definition of the related probability measure.

Being, generally, not orthonormalized, the prospect states do not form a basis in the mind space.

Definition 9. Strategic state

Among the states of the mind space, there exists a special fixed state $|s\rangle \in \mathcal{M}$, playing the role of a reference state, which is termed the *strategic state*. The strategic state of mind is a fixed vector characterizing a particular decision maker, with his/her beliefs, habits, principals, etc., that is, describing

each decision maker as a unique subject. Hence, each space of mind possesses a unique strategic state. Different decision makers possess different strategic states.

Being in the mind space (10), this state can be represented as the decomposition

$$|s\rangle = \sum_n c_n |e_n\rangle \ . \tag{12}$$

Being a unique state, characterizing each decision maker like its fingerprints, it can be normalized to one:

$$\langle s|s\rangle = 1 \ . \tag{13}$$

From Equations (12) and (13), it follows that

$$\sum_n |c_n|^2 = 1.$$

The existence of the strategic state, uniquely defining each particular decision maker, is the principal point distinguishing an *active thinking* quantum system from a *passive* quantum system subject to measurements from an external observer. For a passive quantum system, predictions of the outcome of measurements are performed by summing (averaging) over all possible statistically equivalent states, which can be referred to as a kind of "annealed" situation. In contrast, decisions and observations associated with a thinking quantum system occur in the presence of this unique strategic state, which can be thought of as a kind of fixed "quenched" state. As a consequence, the outcomes of the applications of the quantum-mechanical formalism will thus be different for thinking versus passive quantum systems.

Definition 10. Prospect operators

Each prospect state $|\pi_j\rangle$, together with its Hermitian conjugate $\langle\pi_j|$, defines the *prospect operator*

$$\hat{P}(\pi_j) \equiv |\pi_j\rangle\langle\pi_j| \ . \tag{14}$$

By this definition, the prospect operator is self-adjoint. The family of all prospect operators forms the involutive bijective algebra that is analogous to the algebra of local observables in quantum theory. Since the prospect states, in general, are neither mutually orthogonal nor normalized, the squared operator

$$\hat{P}^2(\pi_j) = \langle\pi_j|\pi_j\rangle\hat{P}(\pi_j)$$

contains the scalar product

$$\langle\pi_j|\pi_j\rangle = \sum_n |a_{nj}|^2 \ ,$$

which does not equal to one. This tells us that the prospect operators, generally, are not idempotent, thus, they are not projection operators. It is only when the prospect is elementary that the related prospect operator

$$\hat{P}(e_n) = |e_n\rangle\langle e_n|$$

becomes idempotent and is a projection operator. But, in general, this is not so.

The properties of the prospect operators follow immediately from those of the prospect states and definition (14). Recall that the prospect operators are analogous to the operators of local observables in

quantum theory. The latter operators are not required to be idempotent. So, the prospect operators are also not required to be such. The intuition of why the prospect operators are not idempotent could be justified by understanding that, in general, a prospect realized twice results in the consequences that are not necessarily the same as a sole prospect realization. For instance, to marry twice is not the same as to marry once.

Definition 11. Prospect probabilities

In quantum theory, the averages over the system state, for the operators from the algebra of local observables, define the observable quantities. In the same way, the averages, over the strategic state, for the prospect operators define the observable quantities, the *prospect probabilities*

$$p(\pi_j) \equiv \langle s|\hat{P}(\pi_j)|s \rangle \ . \tag{15}$$

These are assumed to be normalized to one:

$$\sum_{j=1}^{N_L} p(\pi_j) = 1 \ , \tag{16}$$

where the summation is over all prospects from the prospect lattice (5). By their definition, the quantities (15) are non-negative, since Equation (15) reduces to the modulus of the transition amplitude squared

$$p(\pi_j) = |\langle \pi_j|s \rangle|^2 \ .$$

The normalization in Equation (16) is necessary for the set $\{p(\pi_j)\}$ be the scalar probability measure. In plane words, the fact that all prospects probabilities are summed to one implies that one of them is to be certainly realized.

Definition 12. Utility factor

The diagonal form

$$p_0(\pi_j) \equiv \sum_n \langle s|\hat{P}(e_n)\hat{P}(\pi_j)\hat{P}(e_n)|s \rangle \tag{17}$$

plays the role of the expected utility in classical decision making, justifying its name as the *utility factor*. In order to be generally defined and to be independent of the chosen units of measurement, the utility factor (17) can be normalized as

$$\sum_{j=1}^{N_L} p_0(\pi_j) = 1 \ . \tag{18}$$

The fact that the utility factor (17) is really equivalent to the classical expected utility follows from noticing that

$$\hat{P}(e_n)|s \rangle = c_n|e_n \rangle \ ,$$

hence Equation (17) acquires the form

$$p_0(\pi_j) = \sum_n |c_n|^2 \langle e_n|\hat{P}(\pi_j)|e_n \rangle \ ,$$

where $< e_n|\hat{P}(\pi_j)|e_n >$ plays the role of a utility function, weighted with the probability $|c_n|^2$.

Definition 13. Attraction factor

The nondiagonal term

$$q(\pi_j) \equiv \sum_{m \neq n} \langle s | \hat{P}(e_m) \hat{P}(\pi_j) \hat{P}(e_n) | s \rangle \tag{19}$$

corresponds to the quantum interference effect. Its appearance is typical of quantum mechanics. Such nondiagonal terms do not occur in classical decision theory. This term can be called the *interference factor*. Interpreting its meaning in decision making, we can associate its appearance as resulting from the system deliberation between several alternatives, when deciding which of the latter is more attractive. Thence, the name "attraction factor". Using expansion (12) in Equation (19) yields

$$q(\pi_j) \equiv \sum_{m \neq n} c_m^* c_n \langle e_m | \hat{P}(\pi_j) | e_n \rangle ,$$

which shows that the interference occurs between different elementary prospects in the process of considering a composite prospect π_j. It is worth stressing that the interference factor is nonzero only when the prospect π_j is composite. If it were elementary, say $\pi_j = e_k$ then, since

$$\hat{P}(e_k) | e_n \rangle = \delta_{nk} | e_n \rangle ,$$

we would have

$$q(e_k) = \sum_{m \neq n} c_m^* c_n \delta_{mn} \delta_{nk} = 0 ,$$

and no interference would arise.

Between two prospects, the one which enjoys the larger attraction factor is more attractive.

Definition 14. Prospect ordering

In defining the prospect lattice (5), we have assumed that the prospects could be ordered. Now, after introducing the scalar probability measure, we are in a position to give an explicit prescription for the prospect ordering. We say that the prospect π_1 is *preferable* to π_2 if and only if

$$p(\pi_1) > p(\pi_2) \qquad (\pi_1 > \pi_2) . \tag{20}$$

Two prospects are called indifferent if and only if

$$p(\pi_1) = p(\pi_2) \qquad (\pi_1 = \pi_2) . \tag{21}$$

And the prospect π_1 is preferable or indifferent to π_2 if and only if

$$p(\pi_1) \geq p(\pi_2) \qquad (\pi_1 \geq \pi_2) . \tag{22}$$

These binary relations provide us with an explicit prospect ordering making the prospect set (5) a lattice.

Definition 15. Optimal prospect

Since all prospects in the lattice are ordered, it is straightforward to find among them that one enjoying the largest probability. This defines the *optimal prospect* π_* for which

$$p(\pi_*) \equiv \sup_j p(\pi_j) . \qquad (23)$$

Finding the optimal prospect is the final goal of the decision-making process. Since the prospect probabilities are non-negative, it is possible to find the minimal prospect in the lattice (5) with the smallest probability. And the largest probability defines the optimal prospect π_*. Therefore the prospect set (5) is a complete lattice.

Remark. Generally speaking, all states of the mind space can depend on time t. We do not write explicitly the time dependence, when this makes no difference for the considerations developed below. When this is important, we shall denote the time dependence explicitly.

3. Entangled Prospect States

Prospect states can be of two qualitatively different types.

- A *disentangled prospect state* is a prospect state which is represented as the tensor product of the intention states:

$$|f> = \otimes_i |\psi_i> . \qquad (24)$$

- An *entangled prospect state* is any prospect state that cannot be reduced to the tensor product form of the disentangled prospect states (24).

We define the *disentangled set* as the collection of all admissible disentangled prospect states of form (24):

$$\mathcal{D} \equiv \{|f> = \otimes_i |\psi_i>, \ |\psi_i> \in \mathcal{M}_i\} . \qquad (25)$$

In quantum theory, it is possible to construct various entangled and disentangled states (see, e.g., [80, 81]). In order to explain how entangled states appear in the quantum theory of decision making, let us illustrate the above definitions by an example of a prospect consisting of two intended actions with two mode representations each. Let us consider the prospect of the following two intentions: "to get married" and "to become rich". And let us assume that the intention "to get married" consists of two representations, "to marry A", with the mode state $|A>$, and "to marry B", with the mode state $|B>$. And let the intention "to become rich" be formed by two representations, "to become rich by working hard", with the mode state $|W>$, and "to become rich by being a gangster", with the mode state $|G>$. Thus, there are two intention states of the following type:

$$|\psi_1> = a_1|A> + a_2|B> , \qquad |\psi_2> = b_1|W> + b_2|G> . \qquad (26)$$

The general prospect state has the form

$$|\pi> = c_{11}|AW> + c_{12}|AG> + c_{21}|BW> + c_{22}|BG> , \qquad (27)$$

where the coefficients c_{ij} belong to the field of complex numbers.

Depending on the values of the coefficients c_{ij}, the prospect state (27) can be either disentangled or entangled. If it is disentangled, it must be of the tensor product type (24), which for the present case reads

$$|f> = |\psi_1> \otimes |\psi_2> = a_1 b_1 |AW> + a_1 b_2 |AG> + a_2 b_1 |BW> + a_2 b_2 |BG> . \qquad (28)$$

Both states (27) and (28) include four basic states:

- "to marry A and to work hard", $|AW>$,

- "to marry A and become a gangster", $|AG>$,

- "to marry B and to work hard", $|BW>$,

- "to marry B and become a gangster", $|BG>$.

However, the structure of states (27) and (28) is different. The prospect state (27) is more general and can be reduced to state (28) for special values of the coefficients c_{ij}, but the opposite may not be possible. For instance, the prospect state

$$c_{12}|AG> + c_{21}|BW> , \qquad (29)$$

which is a particular example of state (27) cannot be reduced to any of the states (28), provided that both coefficients c_{12} and c_{21} are non-zero. In quantum mechanics, this state would be called the Einstein-Podolsky-Rosen state, one of the most famous examples of an entangled state [82]. Another example is the prospect state

$$c_{11}|AW> + c_{22}|BG> , \qquad (30)$$

whose quantum-mechanical analog would be called the Bell state [83]. In the case where both c_{11} and c_{22} are non-zero, the Bell state cannot be reduced to any of the states (28) and is thus entangled.

In contrast with the above two examples, the prospect states

$$c_{11}|AW> + c_{12}|AG> , \qquad c_{11}|AW> + c_{21}|BW> ,$$

$$c_{12}|AG> + c_{22}|BG> , \qquad c_{21}|BW> + c_{22}|BG> ,$$

are *disentangled*, since all of them can be reduced to the form (28).

Other examples of *entangled* prospect states are

$$c_{11}|AW> + c_{12}|AG> + c_{21}|BW> , \qquad c_{11}|AW> + c_{12}|AG> + c_{22}|BG> ,$$

$$c_{11}|AW> + c_{21}|BW> + c_{22}|BG> , \qquad c_{12}|AG> + c_{21}|BW> + c_{22}|BG> ,$$

where all coefficients are assumed to be non-zero.

Since the coefficients $c_{ij} = c_{ij}(t)$ are, in general, functions of time, it may happen that a prospect state at a particular time is entangled, but becomes disentangled at another time or, vice versa, a disentangled prospect state can be transformed into an entangled state with changing time [84].

The state of a human being is governed by its physiological characteristics and the available information [85, 86]. These properties are continuously changing in time. Hence the strategic state (12), specific of a person at a given time, may also display temporal evolution, according to different homeostatic processes adjusting the individual to the changing environment [87].

4. Procedure of Decision Making

The process of quantum decision making possesses several features that make it rather different from the classical decision making. These main features are emphasized below.

4.1. *Probabilistic Nature of Decision Making*

Quantum decision making is described as an intrinsically probabilistic procedure. The first step consists in evaluating, consciously and/or subconsciously, the probabilities of choosing different prospects from the point of view of their usefulness and/or appeal to the choosing agent. The strategic state of mind of an agent at some time t is represented by the state $|s>$. Then, the probability of realizing a prospect π_j with the prospect state $|\pi_j>$, under the given strategic state $|s>$, characterizing the agent's state of mind at that time, according to Definition 11, is the *prospect probability* $p(\pi_j)$. The prospect probabilities, defined in (15), possess all the standard probability properties, with the normalization condition (16).

The probabilities are defined in Equation (15) through the prospect states and the strategic state of mind. The latter is normalized to one, according to Equation (13). By their definition, the probabilities are summed to one, as in Equation (16). But the prospect states do not need to be normalized to one, as is stressed in Definition 8. This means that different prospects can have, and usually do have, different weights, corresponding to their different probabilities. In quantum physics, this situation would be similar to defining the cross-section in a scattering experiment over a system containing elementary particles and composite clusters (prospects) formed by these particles.

In the traditional theory of decision making, based on the utility function, the optimal decision corresponds, by definition, to the maximal expected utility which is associated with the maximal anticipated usefulness and profit resulting from the chosen action. In contrast, QDT recognizes that the behavior of an individual is probabilistic, not deterministic. The prospect probability (15) quantifies the probability that a given individual chooses the prospect π_j, under his/her strategic state of mind $|s>$ at a given time t. This translates in experiments into a prediction on the frequency of the decisions taken by an ensemble of subjects under the same conditions. The observed frequencies of different decisions taken by an ensemble of non-interacting subjects making a decision under the same conditions serve as the observable measure of the subjective probability. It is, actually, well-known that subjective probabilities can be calibrated by frequencies or fractions [88, 89].

This specification also implies that the same subject, prepared under the same conditions with the same entangled strategic state of mind $|s>$ at two different times, may choose two different prospects among the same set of prospects, with different relative frequencies determined by the corresponding prospect probabilities (15). Verifying this prediction is a delicate empirical question, because of the possible impact of the "memory" of the past decisions on the next one. In order for the prediction to hold, the two repetitions of the decision process should be independent. Otherwise, the strategic state of mind in the second experiment keeps a memory of the previous choice, which biases the results. This should not be confused with the fact that the projection of the strategic state of mind onto the prospect state π_j, when the decision is made to realize this prospect, ensures that the individual will in general keep his/her decision, whatever it is, when probed a second time sufficiently shortly after the

first decision so that the strategic state of mind, realized just after the projection, has not had time yet to evolve appreciably.

In QDT, the concept of an optimal decision is replaced by a probabilistic decision, when the prospect, which makes $p(\pi_j)$ given by (15) maximal, is the one which corresponds best to the given strategic state of mind. In that sense, the prospect that makes $p(\pi_j)$ maximal can be called optimal with respect to the strategic state of mind. Using the mapping between the subjective probabilities and the frequentist probabilities observed on ensemble of individuals, the prospect that makes $p(\pi_j)$ maximal, will be chosen by more individuals that any other prospect, in the limit of large population sampling sizes. However, other less probable prospects will also be chosen by some smaller subset of the population with frequencies given by the corresponding quantum mechanical probabilities given above.

4.2.   *Entangled Decision Making*

As is explained above, a prospect state $|\pi_j>$ does not have in general the form of the product (24), which means that it is entangled. Therefore, the prospect probability $p(\pi_j)$, generally, cannot be reduced to a product:

$$p(\pi_j) \neq \prod_i p(A_i) \ .$$

In other words, usually the decision making process is naturally entangled.

Consider the example of the specific prospect state (27) associated with the two intentions "to get married" and "to become rich". And suppose that A does not like gangsters, so that it is impossible to marry A and at the same time being a gangster. This implies that the prospect AG cannot be realized, hence $c_{12} = 0$. Assume that B dreams of becoming rich as fast as possible, and a gangster spouse is much more luring for B than a dull person working hard, which implies that $c_{21} = 0$. In this situation, the prospect state (27) reduces to the entangled Bell state $c_{11}|AW> +c_{22}|BG>$. A decision performed under these conditions, resulting in an entangled state, is entangled.

4.3.   *Non-commutativity of Decisions and History Dependence*

There exist numerous real-life examples when decision makers fail to follow their plans and change their mind simply because they experience different outcomes on which their intended plans were based. This change of plans after experiencing particular outcomes is the effect known as dynamic inconsistency [90–92]. In our language [78], this can be considered as a consequence of the non-commutativity of subsequent decisions, resulting from interference and entanglement between intention representations. After studying the effect of interference, we shall give in what follows a rigorous mathematical formulation of the non-commutativity of decisions.

5.   **Interference of Intended Actions**

Interference is the effect that is typical of all those phenomena which are described by wave equations. Following the Bohr's idea [37–40] of describing mental processes in terms of quantum mechanics, one is immediately confronted with the interference effect, since the physical states in quantum mechanics

are characterized by wave functions. The possible occurrence of interference in the problems of decision making has been discussed before on different grounds (see, e.g., [93]). However, no general theory has been suggested, which would explain why and when such a kind of effect would appear, how to predict it, and how to give a quantitative analysis of it that can be compared with empirical observations. In our approach, interference in decision making arises only when one takes a decision involving composite intentions. The corresponding mathematical treatment of these interferences within QDT is presented in the following subsections.

5.1. Illustration of Interference in Decision Making

As an illustration, let us consider the following situation of two intended actions, "to get a friend" and "to become rich". Let the former intention have two mode representations "to get a friend A" and "to get a friend B". And let the second intention also have two representations, "to become rich by working hard" and "to become rich by being a gangster". The corresponding strategic mind state is given by Equation (12), with the evident notation for the basic states $|e_n>$ and the coefficients c_n represented by the identities

$$c_{11} \equiv c_{AW} , \qquad c_{12} \equiv c_{AG} , \qquad c_{21} \equiv c_{BW} , \qquad c_{22} \equiv c_{BG} . \qquad (31)$$

Suppose that one does not wish to choose between these two friends in an exclusive manner, but one hesitates of being a friend to A as well as B, with the appropriate weights. This means that one considers the intended actions A and B, while the way of life, either to work hard or to become a gangster, has not yet been decided.

The corresponding composite prospects

$$\pi_A = A(W + G) , \qquad \pi_B = B(W + G) \qquad (32)$$

are characterized by the prospect states

$$|\pi_A> = a_1|AW> + a_2|AG> , \qquad |\pi_B> = b_1|BW> + b_2|BG> . \qquad (33)$$

Let us stress that the weights correspond to the intended actions, among which the choice is yet to be made. And one should not confuse the intended actions with the actions that have already been realized. One can perfectly deliberate between keeping this or that friend, in the same way, as one would think about marrying A or B in another example above. This means that the choice has not yet been made. And before it is made, there exist deliberations involving stronger or weaker intentions to both possibilities. Of course, one cannot marry both (at least in most Christian communities). But before marriage, there can exist the dilemma between choosing this or that individual.

Calculating the scalar products

$$< \pi_A|s > = a_1^* c_{11} + a_2^* c_{12} , \qquad < \pi_B|s > = b_1^* c_{21} + b_2^* c_{22} , \qquad (34)$$

we find the prospect probabilities.

$$p(\pi_A) = |a_1^* c_{11} + a_2^* c_{12}|^2 , \qquad p(\pi_B) = |b_1^* c_{21} + b_2^* c_{22}|^2 . \qquad (35)$$

Recall that the prospects are characterized by vectors pertaining to the space of mind $\mathcal{M}$, which are not necessarily normalized to one or orthogonal to each other. The main constraint is that the total set of prospect states $\{|\pi_j >\}$ be such that the related probabilities

$$p(\pi_j) \equiv | < \pi_j|s > |^2$$

be normalized to one, according to the normalization condition (16).

The probabilities (35) can be rewritten in another form by introducing the partial probabilities

$$p(AW) \equiv |a_1 c_{11}|^2 \ , \qquad p(AG) \equiv |a_2 c_{12}|^2 \ ,$$
$$p(BW) \equiv |b_1 c_{21}|^2 \ , \qquad p(BG) \equiv |b_2 c_{22}|^2 \ , \tag{36}$$

and the interference terms

$$q(\pi_A) \equiv 2\mathrm{Re}\,(a_1^* c_{11} a_2 c_{12}^*) \ , \qquad q(\pi_B) \equiv 2\mathrm{Re}\,(b_1^* c_{21} b_2 c_{22}^*) \ . \tag{37}$$

Then the probabilities (35) become

$$p(\pi_A) = p(AW) + p(AG) + q(\pi_A) \ , \qquad p(\pi_B) = p(BW) + p(BG) + q(\pi_B) \ . \tag{38}$$

Let us define the *uncertainty angles*

$$\Delta(\pi_A) \equiv \arg\,(a_1^* c_{11} a_2 c_{12}^*) \ , \qquad \Delta(\pi_B) \equiv \arg\,(b_1^* c_{21} b_2 c_{22}^*) \tag{39}$$

and the *uncertainty factors*

$$\varphi(\pi_A) \equiv \cos\Delta(\pi_A) \ , \qquad \varphi(\pi_B) \equiv \cos\Delta(\pi_B) \ . \tag{40}$$

Using these, the interference terms (37) take the form

$$q(\pi_A) = 2\varphi(\pi_A)\,\sqrt{p(AW)p(AG)} \ , \qquad q(\pi_B) = 2\varphi(\pi_B)\,\sqrt{p(BW)p(BG)} \ . \tag{41}$$

The interference terms characterize the existence of deliberations between the decisions of choosing a friend and, at the same time, a type of work.

This example illustrates the observation that the phenomenon of decision interference appears when one considers a composite prospect with several intention representations assumed to be realized simultaneously. Treating a composite prospect as a combination of several sub-prospects, we could consider the global decision as a collection of sub-decisions. Then the arising interference would occur between these sub-decisions. From the mathematical point of view, it appears more convenient to combine several sub-decisions into one global decision and to analyze the interference of different intentions. Thus, we can state that interference in decision making appears only when one decides about a composite prospect.

For the above example of decision making in the case of two intentions, "to get a friend" and "to be rich", the appearance of the interference can be understood as follows. In real life, it is too problematic, and practically impossible, to become a very close friend to several persons simultaneously, since conflict of interests often arises between the friends. For instance, doing a friendly action to one friend may upset or even harm another friend. Any decision making, involving mutual correlations between two persons, necessarily requires taking into account their, sometimes conflicting, interests. This is, actually, one of the origins of the interference in decision making. Another powerful origin of intention interference is the existence of emotions, as will be discussed in the following sections.

5.2. Conditions for Interference Appearance

The situations for which intention interferences is impossible can be classified into two cases, which are examined below. From this classification, we conclude that the necessary conditions for the appearance of intention interferences are that the dimensionality of mind should be not lower than two and that there should be some uncertainty in the considered prospect.

One-dimensional mind

Suppose there are several intended actions $\{A_i\}$, enumerated by the index $i = 1, 2, \ldots$, whose number can be arbitrary. But each intention possesses only a single representation $|A_i >$. Hence, the dimension of "mind" as defined in Definition 7, is $dim\mathcal{M} = 1$. Only a single basic vector exists, which forms the strategic state

$$|s >= |A_1 A_2 \ldots > = \otimes_i |A_i > . \tag{42}$$

In this one-dimensional mind, all prospect states are disentangled, being of the type

$$|\pi >= c\,|A_1 A_2 \ldots > \qquad (|c| = 1) . \tag{43}$$

Therefore, only one probability exists:

$$p = |<\pi|s>|^2 = 1 . \tag{44}$$

Thus, despite the possible large number of arbitrary intentions, they do not interfere, since each of them has just one representation. There can be no intention interference in one-dimensional mind.

Absence of uncertainty

Another important condition for the appearance of intention interference is the existence of uncertainty. To understand this statement, let us consider a given mind with a large dimensionality $dim\mathcal{M} > 1$, characterized by a strategic state $|s >$. Let us analyze a certain prospect with the state

$$|\pi >= c|s > \qquad (|c| = 1) , \tag{45}$$

with an arbitrary strategic state $|s >$. Then again, the corresponding prospect probability is the same as in Equation (44), and no interference can arise.

Thus, the necessary conditions for the appearance of interference are the existence of uncertainty and the dimensionality of mind not lower than 2.

5.3. Interference Alternation

The interference terms, forming the attraction factor (19), enjoy a very important property that can be called the *theorem of interference alternation*.

Theorem 1: *The process of decision making, associated with the prospect probabilities (15) and occurring under the normalization conditions (16) and (18), is characterized by the alternating interference terms, such that the sum of all attraction factors vanishes*:

$$\sum_j q(\pi_j) = 0 . \tag{46}$$

Proof: The proof follows directly from Definitions 13, 15, and 19, taking into account the normalization conditions (16) and (18).

In order to illustrate in more detail the meaning of the above theorem, let us consider a particular case of two intentions, one composing a set $\{A_i\}$ of M_1 representation modes, and another one forming a set $\{X_j\}$ of M_2 modes. The total family of intended actions is therefore

$$\{A_i, X_j \mid i = 1, 2, \ldots, M_1;\ j = 1, 2, \ldots, M_2\} \ .$$

The basis in the mind space is the set $\{|A_i X_j >\}$. The strategic state of mind can be written as an expansion over this basis,

$$|s > = \sum_{ij} c_{ij} |A_i X_j > \ , \tag{47}$$

with the coefficients satisfying the standard normalization

$$\sum_{ij} |c_{ij}|^2 = 1 \ .$$

Let us assume that we are mainly interested in the representation set $\{A_i\}$, while the representations from the set $\{X_j\}$ are treated as secondary. A prospect that is formed of a fixed intention representation A_i, and which can be realized under the occurrence of any of the representations X_j, corresponds to the prospect state

$$|A_i X > = \sum_j \alpha_{ij} |A_i X_j > \ , \tag{48}$$

where $X = \bigcup_j X_j$. The probability of realizing the considered prospect is

$$p(A_i X) \equiv |< A_i X | s >|^2 \ , \tag{49}$$

according to Definition 11.

Following the above formalism of describing the intention interferences, we use the notation

$$p(A_i X_j) \equiv |\alpha_{ij} c_{ij}|^2 \tag{50}$$

for the joint probability of A_i and X_j; and we denote the partial interference terms as

$$q_{jk}(A_i X) \equiv 2\mathrm{Re}\left(\alpha_{ij}^* c_{ij} c_{ik}^* \alpha_{ik}\right) \ . \tag{51}$$

Then, the probability of $A_i X$, given by Equation (48), becomes

$$p(A_i X) = \sum_j p(A_i X_j) + \sum_{j<k} q_{jk}(A_i X) \ . \tag{52}$$

The interference terms appear due to the existence of uncertainty. Therefore, we may define the *uncertainty factor*

$$\varphi_{jk}(A_i X) \equiv \cos \Delta_{jk}(A_i X) \ , \tag{53}$$

where the uncertainty angle is

$$\Delta_{jk}(A_i X) \equiv \arg(\alpha_{ij}^* c_{ij} c_{ik}^* \alpha_{ik}) \ .$$

Then, the interference term (51) takes the form

$$q_{jk}(A_i X) = 2\varphi_{jk}(A_i X) \sqrt{p(A_i X_j) \, p(A_i X_k)} \ . \tag{54}$$

The attraction factor (19) here is nothing but the sum of the interference terms:

$$q(A_i X) \equiv \sum_{j<k} q_{jk}(A_i X) \ . \tag{55}$$

This allows us to rewrite probability (52) as

$$p(A_i X) = \sum_j p(A_i X_j) + q_(A_i X) \ . \tag{56}$$

The joint and conditional probabilities are related in the standard way

$$p(A_i X_j) = p(A_i | X_j) p(X_j) \ . \tag{57}$$

We assume that the family of intended actions is such that at least one of the representations from the set $\{A_i\}$ has to be certainly realized, which means that

$$\sum_i p(A_i X) = 1 \ , \tag{58}$$

and that at least one of the representations from the set $\{X_j\}$ also necessarily happens, that is,

$$\sum_j p(X_j) = 1 \ . \tag{59}$$

Along with these conditions, we keep in mind that at least one of the representations from the set $\{A_i\}$ must be realized for each given X_j, which implies that

$$\sum_i p(A_i | X_j) = 1 \ . \tag{60}$$

Then we immediately come to the equality

$$\sum_i q(A_i X) = 0 \ ,$$

which is just a particular case of the general condition (46).

This equality shows that, if at least one of the terms is non-zero, some of the interference terms are necessarily negative and some are necessarily positive. Therefore, some of the probabilities are depressed, while others are enhanced. This alternation of the interference terms will be shown below to be a pivotal feature providing a clear explanation of the disjunction effect. It is worth emphasizing that the violation of the sure-thing principle, resulting in the disjunction effect, will be shown not to be due simply to the existence of interferences as such, but, more precisely, to the *interference alternation*.

For instance, the depression of some probabilities can be associated with uncertainty aversion, which makes less probable an action under uncertain conditions. In contrast, the probability of other intentions, containing less or no uncertainty, will be enhanced by positive interference terms. This interference alternation is of crucial importance for the correct description of decision making, without which the known paradoxes cannot be explained.

5.4. Less is More

The title of this subsection is taken from a poem of the nineteenth century English poet Robert Browning [94].

In the present context, this expression means that sometimes excessive information is not merely difficult to get, but can even be harmful, resulting in wrong decisions. It often happens that decisions, based on smaller amount of information, are better than those based on larger amount of information. This may happen because, with increasing the amount of information, the choice between alternatives can become more complicated as a result of which uncertainty grows. Increasing complexity often increases uncertainty.

To describe the "less is more" phenomenon in mathematical language, let us consider a prospect π_k^* that is optimal under a fixed information set X_k, with the probability

$$p(\pi_k^*) = p_0(\pi_k^*) + q(\pi_k^*) \ . \tag{61}$$

Suppose, we increase the amount of information by going to the information set X_{k+1}, such that $X_k \in X_{k+1}$, and obtain the related optimal prospect π_{k+1}^*, with the probability

$$p\left(\pi_{k+1}^*\right) = p_0\left(\pi_{k+1}^*\right) + q\left(\pi_{k+1}^*\right) \ . \tag{62}$$

Assume that the utilities of these two prospects are the same,

$$p_0\left(\pi_{k+1}^*\right) = p_0(\pi_k^*) \ , \tag{63}$$

while the uncertainty in the decision making process increases, so that the attraction factor decreases,

$$q\left(\pi_{k+1}^*\right) \ < \ q(\pi_k^*) \ . \tag{64}$$

Then, the relation between the prospect probabilities

$$p(\pi_k^*) - p\left(\pi_{k+1}^*\right) = q(\pi_k^*) - q\left(\pi_{k+1}^*\right) \ > \ 0 \tag{65}$$

tells us that the decision process leading to choosing prospect π_k^* is clearer than for prospect π_{k+1}^*, because the larger value of the corresponding probability makes the signal stronger for the decision maker, resulting in a larger frequency of choices π_k^*. As the information set is increased, in the presence of many alternatives, the preferred prospect becomes less clearly defined as the top choice. As a consequence, a lack of efficiency, a growing indeterminacy and ultimately the freezing of the decision process can ensue.

When dealing with complex nonlinear problems, excessive information can lead to incorrect conclusions because of the extreme sensitivity of nonlinear problems to minor details. As simple examples, when excessive information can be harmful, we may mention the following typical cases from physics.

Example 1. How to describe the state of air in a room? The unreasonable decision would be to analyze the motion of all molecules in the room, specifying all their interactions, positions and velocities. Such a decision would lead to not merely extremely overcomplicated calculations, but even can result in

incorrect conclusions. The reasonable decision is to characterize the state of the air by defining the room temperature, volume, and atmospheric pressure.

Example 2. How to characterize the water flow in a river? A silly decision would be to consider the motion of all water molecules in the river describing their locations, velocities, interactions, and so on. Contrary to this, a clever decision is to use the hydrodynamic equations.

Example 3. How to describe a large social system? Again, the unreasonable decision would be to collect all possible information on each member of the society. Then, being overloaded by senseless information, one would be lost in secondary details, being unable to make any clever conclusion. Instead of this, it is often (though may be not always) sufficient to consider the society composed of typical (or "representative") agents.

6. Disjunction Effect

The disjunction effect was first specified by Savage [62] as a violation of the "sure-thing principle", which can be formulated as follows: *If the alternative A is preferred to the alternative B, when an event X_1 occurs, and it is also preferred to B, when an event X_2 occurs, then A should be preferred to B, when it is not known which of the events, either X_1 or X_2, has occurred.*

6.1. Sure-Thing Principle

Let us now show how the sure-thing principle arises in classical probability theory.

Let us consider a field of events $\{A, B, X_j | j = 1, 2, \ldots\}$ equipped with the classical probability measures [95]. We denote the classical probability of an event A by the capital letter $P(A)$ in order to distinguish it from the probability $p(A)$ defined in the previous sections by means of quantum rules. We shall denote, as usual, the conditional probability of A under the knowledge of X by $P(A|X)$ and the joint probability of A and X, by $P(AX)$. We assume that at least one of the events X_j from the set $\{X_j\}$ certainly happens and that the X_i are mutually exclusive and exhaustive, which implies that

$$\sum_j P(X_j) = 1 \ . \tag{66}$$

The probability of A, when X_j is not specified, that is, when at least one of X_j happens, is denoted by $P(AX)$, where $X = \bigcup_j X_j$. The same notations are applied to B. Following the common wisdom, we understand the statement "A is preferred to B" as meaning that $P(AX) > P(BX)$. Then the following theorem is valid.

Theorem 2: *If for all $j = 1, 2, \ldots$, one has*

$$P(A|X_j) > P(B|X_j) \ , \tag{67}$$

then

$$P(AX) > P(BX) \ . \tag{68}$$

Proof: It is straightforward that, under $X = \bigcup_j X_j$, one has

$$P(AX) = \sum_j P(AX_j) = \sum_j P(A|X_j)P(X_j) \tag{69}$$

and

$$P(BX) = \sum_j P(BX_j) = \sum_j P(B|X_j)P(X_j) \ . \tag{70}$$

From Equations (69) and (70), under assumption (67), inequality (68) follows immediately.

The above proposition is a theorem of classical probability theory. Savage [62] proposed to use it as a normative statement on how human beings make consistent decisions under uncertainty. As such, it is no more a theorem but a testable assumption about human behavior. In other words, empirical tests showing that humans fail to obey the sure-thing principle must be interpreted as a failure of humans to abide to all the rules of classical probability theory.

6.2. *Disjunction-Effect Examples*

Thus, according to standard classical probability theory which is held by most statisticians as the only rigorous mathematical description of risks, and therefore as the normative guideline describing rational human decision making, the sure-thing principle should be always verified in empirical tests involving real human beings. However, numerous violations of this principle have been investigated empirically [62, 96–99]. In order to be more specific, let us briefly outline some examples of the violation of the sure-thing principle, referred to as the disjunction effect.

(i) *To gamble or not to gamble*?

A typical setup for illustrating the disjunction effect is a two-step gamble [96]. Suppose that a group of people accepted a gamble, in which the player can either win (X_1) or lose (X_2). After one gamble, they are invited to gamble a second time, being free to either accept the second gamble (A) or to refuse it (B). Experiments by Tversky and Shafir [96] showed that the majority of people accept the second gamble when they know the result of the first one, in any case, whether they won or lost in the previous gamble. In the language of conditional probability theory, this translates into the fact that people act as if $P(A|X_1)$ is larger than $P(B|X_1)$ and $P(A|X_2)$ is larger than $P(B|X_2)$ as in Equation (67). At the same time, it turns out that the majority refuses to gamble the second time when the outcome of the first gamble is not known. The second empirical fact implies that people act as if $P(BX)$ overweighs $P(AX)$, in blatant contradiction with inequality (68), which should hold according to the theorem resulting from (67). Thus, a majority accepted the second gamble after having won or lost in the first gamble, but only a minority accepted the second gamble when the outcome of the first gamble was unknown to them. This provides an unambiguous violation of the Savage sure-thing principle.

(ii) *To buy or not to buy*?

Another example, studied by Tversky and Shafir [96], had to do with a group of students who reported their preferences about buying a non-refundable vacation, following a tough university test. They could pass the exam (X_1) or fail (X_2). The students had to decide whether they would go on vacation (A) or abstain (B). It turned out that the majority of students purchased the vacation when they passed the exam as well as when they had failed, so that condition (67) was valid. However, only a minority

of participants purchased the vacation when they did not know the results of the examination. Hence, inequality (68) was violated, demonstrating again the disjunction effect.

(iii) *To sell or not to sell*?

The stock market example, analyzed by Shafir and Tversky [100], is a particularly telling one, involving a deliberation taking into account a future event, and not a past one as in the two previous cases. Suppose we consider the USA presidential election, when either a Republican wins (X_1) or a Democrat wins (X_2). On the eve of the election, market players can either sell certain stocks from their portfolio (A) or hold them (B). It is known that a majority of people would be inclined to sell their stocks, if they would know who wins, regardless of whether the Republican or Democrat candidate wins the upcoming election. This is because people expect the market to fall after the elections. Hence, condition (67) is again valid. At the same time, a great many people do not sell their stocks before knowing who really won the election, thus contradicting the sure-thing principle and the inequality (68). Thus, investors could have sold their stocks before the election at a higher price but, obeying the disjunction effect, they were waiting until after the election, thereby selling at a lower price after stocks have fallen. Many market analysts believe that this is precisely what happened after the 1988 presidential election, when George Bush defeated Michael Dukakis.

There are plenty of other more or less complicated examples of the disjunction effect [62, 96–98, 100–102]. The common necessary conditions for the disjunction effect to arise are as follows. First, there should be several events, each characterized by several alternatives, as in the two-step gambles. Second, there should necessarily exist some uncertainty, whether with respect to the past, as in the examples (i) and (ii), or with respect to the future, as in the example (iii).

Several ways of interpreting the disjunction effect have been analyzed. Here, we do not discuss the interpretations based on the existence of some biases, such as the gender bias, or the bias invoking the notion of decision complexity, which have already been convincingly ruled out [97, 103]. We describe the reason-based explanation which appears to enjoy a wide-spread following and discuss its limits before turning to the view point offered by QDT.

6.3. Reason-Based Analysis

The dominant approach for explaining the disjunction effect is the reason-based analysis of decision making [96, 97, 100, 102, 104]. This approach explains choice in terms of the balance between reasoning for and against the various alternatives. The basic intuition is that when outcomes are known, a decision maker may easily come up with a definitive reason for choosing an option. However, in the case of uncertainty, when the outcomes are not known, people may lack a clear reason for choosing an option and consequently they abstain and make an irrational choice.

From our perspective, the weakness of the reason-based analysis is that the notion of "reason" is too vague and subjective. Reasons are not only impossible to quantify, but it is difficult, if possible at all, to give a qualitative definition of what they are. Consider example (i) "to gamble or not to gamble?" Suppose you have already won at the first step. Then, you can rationalize that gambling a second time is not very risky: if you now loose, this loss will be balanced by the first win on which you were not counting anyway, so that you may actually treat it differently from the rest of your wealth, according

to the so-called "mental accounting" effect; and if you win again, your profit will be doubled. Thus, you have a "reason" to justify the attractiveness of the second gamble. But, it seems equally justified to consider the alternative "reason": if you have won once, winning the second time may seem less probable (the so-called gambler's fallacy), and if you loose, you will keep nothing of your previous gain. This line of reasoning justifies to keep what you already got and to forgo the second gamble.

Suppose now you have lost in the first gamble and know it. A first reasoning would be that the second gamble offers a possibility of getting out of the loss, which provides a reason for accepting the second gamble. However, you may also think that the win is not guaranteed, and your situation could actually worsen, if you loose again. Therefore, this makes it more reasonable not to risk so much and to refrain from the new gamble.

Consider now the situation where you are kept ignorant of whether you have won or lost in the first gamble. Then, you may think that there is no reason and therefore no motivation for accepting the second gamble, which is the standard reason-based explanation. But, one could argue that it would be even more logical if you would think as follows: Okay, I do not know what has happened in the first gamble. So, why should I care about it? Why don't I try again my luck? Certainly, there is a clear reason for gambling that could propagate the drive to gamble a second time.

This discussion is not pretending to demonstrate anything other than that the reason-based explanation is purely ad-hoc, with no real explanatory power; it can be considered in a sense as a reformulation of the disjunction fallacy. It is possible to multiply the number of examples demonstrating the existence of quite "reasonable" justifications for doing something as well as a reason for just doing the opposite. It seems to us that the notion of "reason" is not well defined and one can always invent in this way a justification for anything. Thus, we propose that the disjunction effect has no direct relation to reasoning. In the following section, we suggest another explanation of this effect based on QDT, specifically the interference between the two uncertain outcomes resulting from an aversion to uncertainty (uncertainty-aversion principle), which provides a *quantitative* testable prediction.

6.4. Quantitative Analysis Within Quantum Decision Theory

The disjunction effect, described above, finds a natural explanation in the frame of the Quantum Decision Theory, as is shown below.

Application to Disjunction-Effect Examples

The possibility of connecting the violation of the sure-thing principle with the occurrence of interference has been mentioned in several articles (see, e.g., [93]). But all these attempts were ad hoc assumptions not based on a self-consistent theory. Our explanation of the disjunction effect differs from these attempts in several aspects. First, we consider the disjunction effect as just one of several possible effects in the frame of the *general theory*. The explanation is based on the theorem of *interference alternation*, which has never been mentioned, but without which no explanation can be complete and self-consistent. We stress the importance of the *uncertainty-aversion principle*. Also, we offer a *quantitative estimate* for the effect, which is principally new.

Let us discuss the two first examples illustrating the disjunction effect, in which the prospect consists of two intentions with two representations each. One intention "to decide about an action" has the

representations "to act" (A) and "not to act" (B). The second intention "to know the results" (or "to have information") has also two representations. One (X_1) can be termed "to learn about the win" (gamble won, exam passed), the other (X_2) can be called "to learn about the loss" (gamble lost, exam failed). With the numbers of these representations $M_1 = 2$ and $M_2 = 2$, the dimension of mind, given in Definition 7, is $dim\mathcal{M} = M_1 M_2 = 4$.

For the considered cases, the general set of Equations (56) reduces to two equations

$$p(AX) = p(AX_1) + p(AX_2) + q(AX) \,,$$

$$p(BX) = p(BX_1) + p(BX_2) + q(BX) \,, \tag{71}$$

in which again $X = \bigcup_j X_j$ and the interference terms are the attraction factors

$$q(AX) = 2\varphi(AX) \sqrt{p(AX_1)\, p(AX_2)} \,,$$

$$q(BX) = 2\varphi(BX) \sqrt{p(BX_1)\, p(BX_2)} \,. \tag{72}$$

Of course, Equations (71) and (72) could be postulated, but then it would not be clear where they come from. In QDT, these equations appear naturally. Here $\varphi(AX)$ and $\varphi(BX)$ are the uncertainty factors defined in (53). The normalizations (58) and (59) become

$$p(AX) + p(BX) = 1 \,, \qquad p(X_1) + p(X_2) = 1 \,. \tag{73}$$

The normalization condition (60) gives

$$p(A|X_1) + p(B|X_1) = 1 \,, \qquad p(A|X_2) + p(B|X_2) = 1 \,. \tag{74}$$

The uncertainty factors can be rewritten as

$$\varphi(AX) = \frac{q(AX)}{2\sqrt{p(AX_1)p(AX_2)}} \,, \qquad \varphi(BX) = \frac{q(BX)}{2\sqrt{p(BX_1)p(BX_2)}} \,, \tag{75}$$

with the interference terms being

$$q(AX) = p(AX) - p(AX_1) - p(AX_2) \,, \qquad q(BX) = p(BX) - p(BX_1) - p(BX_2) \,. \tag{76}$$

The principal point is the condition of *interference alternation* (46), which now reads

$$q(AX) + q(BX) = 0 \,. \tag{77}$$

Without this condition (77), the system of equations for the probabilities would be incomplete, and the disjunction effect could not be explained.

In the goal of explaining the disjunction effect, it is not sufficient to merely state that some type of interference is present. It is necessary to determine (quantitatively if possible) why the probability of acting is suppressed, while that of remaining passive is enhanced. Our aim is to evaluate the expected size and signs of the interference terms $q(AX)$ (for acting under uncertainty) and $q(BX)$ (for remaining inactive under uncertainty). Obviously, it is an illusion to search for a universal value that everybody will use. Different experiments with different people have indeed demonstrated a significant heterogeneity

among people, so that, in the language of QDT, this means that the values of the interference terms can fluctuate from individual to individual. A general statement should here refer to the behavior of a sufficiently large ensemble of people, allowing us to map the observed frequentist distribution of decisions to the predicted QDT probabilities.

Attraction Factors as Interference Terms

The interference terms (72) can be rewritten as

$$q(AX) = 2\varphi(AX) \sqrt{p(A|X_1)p(X_1)p(A|X_2)p(X_2)} \, ,$$

$$q(BX) = 2\varphi(BX) \sqrt{p(B|X_1)p(X_1)p(B|X_2)p(X_2)} \, . \tag{78}$$

The interference-alternation theorem (Theorem 1), which leads to Equation (77), implies that

$$|q(AX)| = |q(BX)| \, , \tag{79}$$

and

$$\text{sign}[\varphi(AX)] = -\text{sign}[\varphi(BX)] \, . \tag{80}$$

Hence, in the case where $p(A|X_j) > p(B|X_j)$, which is characteristic of the examples illustrating the disjunction effect, one must have the uncertainty factors which exhibit the opposite property, $|\varphi(AX)| < |\varphi(BX)|$, so as to compensate the former inequality to ensure the validity of the equality (79) for the absolute values of the interference terms. The next step is to determine the sign of $\varphi(AX)$ (and thus of $\varphi(BX)$) from (80) and their typical amplitudes $|\varphi(AX)|$ and $|\varphi(BX)|$.

Signs of Uncertainty Factors

A fundamental well-documented characteristic of human beings is their aversion to uncertainty, *i.e.*, the preference for known risks over unknown risks [105]. As a consequence, the propensity/utility (and therefore the probability) to act under larger uncertainty is smaller than under smaller uncertainty. Mechanically, this implies that it is possible to specify the sign of the uncertainty factors, yielding

$$\text{sign}[\varphi(AX)] = -\text{sign}[\varphi(BX)] < 0 \, , \tag{81}$$

since A (respectively B) refers to acting (respectively, to remain inactive).

Amplitudes of Uncertainty Factors

As a consequence of Equation (81) and also of their mathematical definition (53), the uncertainty factors vary in the intervals

$$-1 \le \varphi(AX) \le 0 \, , \qquad 0 \le \varphi(BX) \le 1 \, . \tag{82}$$

Without any other information, the simplest prior is to assume a uniform distribution of the uncertainty factors in each interval, so that their expected values are respectively

$$\overline{\varphi}(AX) = -\frac{1}{2} \, , \qquad \overline{\varphi}(BX) = \frac{1}{2} \, . \tag{83}$$

Choosing in that way the average values of the uncertainty factors is equivalent to using a representative agent, while the general approach is fully taking into account a pre-existing heterogeneity. That is, the values (83) should be treated as estimates for the expected uncertainty factors, corresponding to these factors averaged with the uniform distribution over the large number of agents.

Interference-Quarter Law

To complete the calculation of $q(AX)$ and of $q(BX)$ given by Equations (78), we also assume the non-informative uniform prior for all probabilities appearing below the square-roots, so that their expected values are all $1/2$ since they vary between 0 and 1. Using these in Equation (78) results in the interference-quarter law

$$\overline{q}(AX) = -0.25 \,, \qquad \overline{q}(BX) = 0.25 \,, \tag{84}$$

valid for the four-dimensional mind composed of two intentions with two representations each.

As a consequence, the probabilities for acting or for remaining inactive under uncertainty, given by Equations (71), can be evaluated as

$$p(AX) = p(AX_1) + p(AX_2) - 0.25 \,,$$

$$p(BX) = p(BX_1) + p(BX_2) + 0.25 \,. \tag{85}$$

The influence of intention interference, in the presence of uncertainty, on the decision making process at the basis of the disjunction effect can thus be estimated a priori. The sign of the effect is controlled by the aversion to uncertainty exhibited by people (uncertainty-aversion principle). The amplitude of the effect can be estimated, as shown above, from simple priors applied to the mathematical structure of the QDT formulation.

Uncertainty-Aversion Principle

The above calculation implies that the disjunction effect can be interpreted as essentially an emotional reaction associated with the *aversion to uncertainty*. An analogy can make the point: it is widely recognized that uncertainty frightens living beings, whether humans or animals. It is also well documented that fear paralyzes, as in the cartoon of the "rabbit syndrome," when a rabbit stays immobile in front of an approaching boa instead of running away. There are many circumstantial evidences that uncertainty may frighten people as a boa frightens rabbits. Being afraid of uncertainty, a majority of human beings may be hindered to act. In the presence of uncertainty, they do not want to act, so that they refuse the second gamble, as in example (i), or forgo the purchase of a vacation, as in example (ii), or refrain from selling stocks, as in example (iii). Our analysis suggests that it is the aversion to uncertainty that paralyzes people and causes the disjunction effect.

It has been reported that, if people, when confronting uncertainty paralyzing them against acting, are presented with a detailed explanation of the possible outcomes, they then may change their mind and decide to act, thus reducing the disjunction effect [96, 97]. Thus, by encouraging people to think by providing them additional explanations, it is possible to influence their minds. In such a case, reasoning plays the role of a kind of therapeutic treatment decreasing the aversion to uncertainty. This line of reasoning suggests that it should be possible to decrease the aversion to uncertainty by other means,

perhaps by distracting them or by taking food, drink or drug injections. This provides the possibility to test for the dependence of the strength of the disjunction effect with respect to various parameters which may modulate the aversion response of individuals to uncertainty.

We should stress that our explanation departs fundamentally from the standard reason-based rationalization of the disjunction effect summarized above. Rather than using what we perceive is an hoc explanation, we anchor the disjunction effect on the very fundamental characteristic of living beings, that of the aversion to uncertainty. This allows us to construct a robust and parsimonious explanation. But this explanation arises only within QDT, because the latter allows us to account for the complex emotional, often subconscious, feelings as well as many unknown states of nature that underlie decision making. Such unknown states, analogous to hidden variables in quantum mechanics, are taken into account by the formalism of QDT through the interference alternation effect, capturing mental processes by means of quantum-theory techniques.

It is appropriate here to remind once more that it was Bohr who advocated throughout all his life the idea that mental processes do bear close analogies with quantum processes (see, e.g., [37–40]). Since interference is one of the most striking characteristic features of quantum processes, the analogy suggests that it should also arise in mental processes as well. The existence of interference in decision making disturbs the classical additivity of probabilities. Indeed, we take as an evidence of this the nonadditivity of probabilities in psychology which has been repeatedly observed [106–108], although it has not been connected with interference.

Numerical Analysis of Disjunction Effect

In the frame of QDT, it is possible, not merely to connect the existence of the disjunction effect with interference, but to give *quantitative predictions*. Below, this is illustrated by the numerical explanation of the examples described above.

(i) *To gamble or not to gamble*?

Let us turn to the example of gambling. The statistics reported by Tversky and Shafir [96] are

$$p(A|X_1) = 0.69 \,, \qquad p(A|X_2) = 0.59 \,, \qquad p(AX) = 0.36 \,.$$

Then Equations (73) and (74) give

$$p(B|X_1) = 0.31 \,, \qquad p(B|X_2) = 0.41 \,, \qquad p(BX) = 0.64 \,.$$

Recall that the disjunction effect here is the violation of the sure-thing principle, so that, although $p(A|X_j) > p(B|X_j)$ for $j = 1, 2$, one observes nevertheless that $p(AX) < p(BX)$. In the experiment reported by Tversky and Shafir [96], the probabilities for winning or for losing were identical: $p(X_1) = p(X_2) = 0.5$. Then, using relation (57), we obtain

$$p(AX_1) = 0.345 \,, \qquad p(AX_2) = 0.295 \,, \qquad p(BX_1) = 0.155 \,, \qquad p(BX_2) = 0.205 \,.$$

For the interference terms, we find

$$q(AX) = -0.28 \,, \qquad q(BX) = 0.28 \,. \tag{86}$$

The uncertainty factors (75) are therefore

$$\varphi(AX) = -0.439 \,, \qquad \varphi(BX) = 0.785 \,.$$

They are of opposite sign, in agreement with condition (80). The probability $p(AX)$ of gambling under uncertainty is suppressed by the negative interference term $q(AX) < 0$. Reciprocally, the probability $p(BX)$ of not gambling under uncertainty is enhanced by the positive interference term $q(BX) > 0$. This results in the disjunction effect, when $p(AX) < p(BX)$.

It is important to stress that the observed amplitudes in (86) are close to the interference-quarter law (84). Actually, within the experimental accuracy with a statistical error about 20%, the found interference terms cannot be distinguished from the value 0.25. Thus, even not knowing the results of the considered experiment, we are able to *quantitatively* predict the strength of the disjunction effect.

(ii) *To buy or not to buy*?

For the second example of the disjunction effect, the data, taken from [96], read

$$p(A|X_1) = 0.54 \,, \qquad p(A|X_2) = 0.57 \,, \qquad p(AX) = 0.32 \,.$$

Following the same procedure as above, we get

$$p(B|X_1) = 0.46 \,, \qquad p(B|X_2) = 0.43 \,, \qquad p(BX) = 0.68 \,.$$

Given again that the two alternative outcomes are equiprobable, $p(X_1) = p(X_2) = 0.5$, we find

$$p(AX_1) = 0.270 \,, \qquad p(AX_2) = 0.285 \,, \qquad p(BX_1) = 0.230 \,, \qquad p(BX_2) = 0.215 \,.$$

For the interference terms, we obtain

$$q(AX) = -0.235 \,, \qquad q(BX) = 0.235 \,. \tag{87}$$

The uncertainty factors are

$$\varphi(AX) = -0.424 \,, \qquad \varphi(BX) = 0.528 \,.$$

Again, the values obtained in (87) are close to our predicted interference-quarter law (84). More precisely, these values are actually undistinguished from 0.25 within the statistical error 20%, typical of the discussed experiments.

Because of the uncertainty aversion, the probability $p(AX)$ of purchasing a vacation is suppressed by the negative interference term $q(AX) < 0$. At the same time, the probability $p(BX)$ of not buying a vacation under uncertainty is enhanced by the positive interference term $q(BX) > 0$. This alternation of interferences causes the disjunction effect resulting in $p(AX) < p(BX)$. It is necessary to stress it again that without this interference alternation no explanation of the disjunction effect is possible in principle.

In the same way, our approach can be applied to any other situation related to the disjunction effect associated with the violation of the sure-thing principle. We now turn to another deviation from rational decision making, known under the name of the conjunction fallacy.

7. Conjunction Fallacy

The conjunction fallacy constitutes another example revealing that intuitive estimates of probability by human beings do not conform to the standard probability calculus. This effect was first studied by Tversky and Kahneman [109, 110] and then discussed in many other works (see, e.g., [101, 111–114]). Despite an extensive debate and numerous attempts to interpret this effect, there seems to be no consensus on the origin of the conjunction fallacy [114].

Here, we show that this effect finds a natural explanation in QDT. It is worth emphasizing that we do not invent a special scheme for this particular effect, but we show that it is a natural consequence of the general theory we have developed. In order to claim to explain the conjunction fallacy in terms of an interference effect in a quantum description of probabilities, it is necessary to derive the quantitative values of the interference terms, amplitudes and signs, as we have done above for the examples illustrating the disjunction effect. This has never been done before. Our QDT provides the necessary ingredients, in terms of the uncertainty-aversion principle, the theorem on interference alternations, and the interference-quarter law. Only the establishment of these general laws can provide an explanation of the conjunction fallacy, that can be taken as a positive step towards validating QDT, according to the general methodology of validating theories [115]. Finally, in our comparison with available experimental data, we analyze a series of experiments and demonstrate that all their data substantiate the validity of the general laws of the theory.

7.1. Conjunction Rule

Let us first briefly recall the conjunction rule of standard probability theory. Let us consider an event A that can occur together with another one among several other events X_j, where $j = 1, 2, \ldots$. The probability of an event, estimated within classical probability theory, is again denoted with the capital letter $P(A)$, to distinguish it from the probability $p(A)$ in our quantum approach. According to standard probability theory [95], one has

$$P(AX) = \sum_j P(AX_j) \,. \tag{88}$$

Since all terms in the sum (88) are positive, the conjunction rule tells us that

$$P(AX) \geq P(AX_j) \qquad (\forall\, j\,). \tag{89}$$

That is, the probability for the occurrence of the conjunction of two events is never larger than the probability for the occurrence of a separate event.

7.2. Conjunction Error

Counterintuitively, humans rather systematically violate the conjunction rule (89), commonly making statements such that

$$p(AX) < p(AX_j) \,, \tag{90}$$

for some j, which is termed the *conjunction fallacy* (Tversky and Kahneman [109, 110]). The difference

$$\varepsilon(AX_j) \equiv p(AX_j) - p(AX) \tag{91}$$

is called the *conjunction error*, which is positive under conditions in which the conjunction fallacy is observed.

A typical situation is when people judge about a person, who can possess a characteristic A and also some other characteristics X_j. This, e.g., can be "possessing a trait" or "not having the trait", since not having a trait is also a characteristic. The often-cited example of Tversky and Kahneman [109] is as follows: "Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations. Which is more likely? (i) Linda is a bank teller; (ii) Linda is a bank teller and is active in the feminist movement." Most people answer (ii) which is an example of the conjunction fallacy (90).

Numerous other examples of the fallacy are described in the literature [101, 110–114]. It is important to stress that this fallacy has been reliably and repeatedly documented, that it cannot be explained by the ambiguity of the word "likely" used in the formulation of the question, and that it appears to involve a failure to coordinate the logical structure of events in the presence of chance [114]. The conjunction fallacy cannot be explained by prospect theory [116] and also remains when different bracketing effects [117–121] are taken into account.

7.3. Conjunction Interference

Within QDT, the conjunction fallacy finds a simple and natural explanation. Let us consider a typical situation of the fallacy, when one judges a person who may have a characteristic A, treated as primary, and who may also possess, or not possess, another characteristic, labeled as secondary. Generally, the person could also be an object, a fact, or anything else, which could combine several features. Translating this situation to the mathematical language of QDT, we see that it involves two intentions. One intention, with just one representation, is "to decide whether the object has the feature A." The second intention "to decide about the secondary feature" has two representations, when one decides whether "the object has the special characteristic" (X_1) or "the object does not have this characteristic" (X_2).

For these definitions, and following the general scheme, we have

$$p(AX) = p(AX_1) + p(AX_2) + q(AX) = p(A|X_1)p(X_1) + p(A|X_2)p(X_2) + q(AX) \ . \tag{92}$$

This is a typical situation where a decision is taken under uncertainty. The uncertainty-aversion principle requires that the interference term $q(AX)$ should be negative ($q(AX) < 0$). Indeed, this reflects that the probability for a human to act under larger uncertainty is smaller than under smaller uncertainty, in line with definition (75) and condition (81).

Taking the perspective of the representation X_1, definition (91) together with Equations (92) imply that the conjunction error reads

$$\varepsilon(AX_1) = |q(AX)| - p(AX_2) \ . \tag{93}$$

The condition for the conjunction fallacy to occur is that the error (93) be positive, which requires that the interference term be sufficiently large, such that the *conjunction-fallacy condition*

$$|q(AX)| > p(AX_2) \tag{94}$$

be satisfied.

The QDT thus predicts that a person will make a decision exhibiting the conjunction fallacy when (i) uncertainty is present and (ii) the interference term, which is negative by the uncertainty-aversion principle, has a sufficiently large amplitude, according to condition (94).

7.4. Comparison with Experiments

For a quantitative analysis, we take the data from Shafir *et al.* [101], who present one of the most carefully accomplished and thoroughly discussed set of experiments. Shafir *et al.* questioned large groups of students in the following way. The students were provided with booklets each containing a brief description of a person. It was stated that the described person could have a primary characteristic (A) and could have additionally a second characteristic (X_1), or could be free of this second characteristic (X_2).

In total, there were 28 experiments separated into two groups according to the conjunctive category of the studied characteristics. In 14 cases, the features A and X_1 were compatible with each other, and in the other 14 cases, they were incompatible. The characteristics were treated as compatible, when they were felt as closely related according to some traditional wisdom, for instance, "woman teacher" (A) and "feminist" (X_1). Another example of compatible features is "chess player" (A) and "professor" (X_1). Those characteristics that were not related by direct logical connections were considered as incompatible, such as "bird watcher" (A) and "truck driver" (X_1) or "bicycle racer" (A) and "nurse" (X_1).

In each of the 28 experiments, the students were asked to evaluate both the typicality and the probability of A and AX_1. Since normal people usually understand "typicality" just as a synonym of probability, and vice versa, the prediction on typicality were equivalent to estimates of probabilities. This amounts to considering only how the students estimated the probability $p(AX)$, with $X = X_1 + X_2$, that the considered person possesses the stated primary feature and the probability $p(AX_1)$ that the person has both characteristics A and X_1.

An important quality of the experiments by Shafir *et al.* [101] lies in the large number of tests which were performed. Indeed, a given particular experiment is prone to exhibit a significant amount of variability, randomness or "noise". Not only the interrogated subjects exhibited significant idiosyncratic differences, with diverse abilities, logic, and experience, but, in addition, the questions were quite heterogeneous. Even the separation of characteristics into two categories of compatible and incompatible pairs is to a great extent arbitrary. Consequently, no one particular case provides a sufficiently clear-cut conclusion on the existence or absence of the conjunction effect. It is only by realizing a large number of interrogations, with a variety of different questions, and by then averaging the results, that it is possible to make justified conclusions on whether or not the conjunction fallacy exists. The set of experiments performed by Shafir *et al.* [101] well satisfies these requirements.

For the set of compatible pairs of characteristics, it turned out that the average probabilities were $p(AX) = 0.537$ and $p(AX_1) = 0.567$, with statistical errors of 20%. Hence, within this accuracy,

$p(AX)$ and $p(AX_1)$ coincide and no conjunction fallacy arises for compatible characteristics. From the view point of QDT, this is easily interpreted as due to the lack of uncertainty: since the features A and X_1 are similar to each other, one almost certainly yielding the other, there is no uncertainty in deciding, hence, no interference, and, consequently, no conjunction fallacy.

However, for the case of incompatible pairs of characteristics, the situation was found to be drastically different. To analyze the related set of experiments, we follow the general scheme of the previous subsection, using the same notations. We have the prospect with two intentions, one intention is to evaluate a primary feature (A) of the object, and another intention is to decide whether, at the same time, the object possesses a secondary feature (X_1) or does not possess it (X_2). Taking the data for $p(X_j)$ and $p(AX_1)$ from Shafir *et al.* [101], we calculate $q(AX)$ for each case separately and then average the results. In the calculations, we take into account that the considered pairs of characteristics are incompatible with each other. The simplest and most natural mathematical embodiment of the property of "incompatibility" is to take the probabilities of possessing A, under the condition of either having or not having X_1, as equal, that is, $p(A|X_j) = 0.5$. For such a case of incompatible pairs of characteristics, Equation (92) reduces to

$$p(AX) = \frac{1}{2} + q(AX) \,. \tag{95}$$

The results, documenting the existence of the interference terms underlying the conjunction fallacy, are presented in Table 1, which gives the abbreviated names for the object characteristics, whose detailed description can be found in Shafir *et al.* [101].

Table 1. Conjunction fallacy and related interference terms caused by the decision under uncertainty. The average interference term is in good agreement with the interference-quarter law. The empirical data are taken from Shafir *et al.* [101].

	characteristics	$p(AX)$	$p(AX_1)$	$q(AX)$
A	bank teller	0.241	0.401	-0.259
X_1	feminist			
A	bird watcher	0.173	0.274	-0.327
X_1	truck driver			
A	bicycle racer	0.160	0.226	-0.340
X_1	nurse			
A	drum player	0.266	0.367	-0.234
X_1	professor			
A	boxer	0.202	0.269	-0.298
X_1	chef			
A	volleyboller	0.194	0.282	-0.306
X_1	engineer			
A	librarian	0.152	0.377	-0.348
X_1	aerobic trainer			

Table 1 *Cont.*

A	hair dresser	0.188	0.252	-0.312
X_1	writer			
A	floriculturist	0.310	0.471	-0.190
X_1	state worker			
A	bus driver	0.172	0.314	-0.328
X_1	painter			
A	knitter	0.315	0.580	-0.185
X_1	correspondent			
A	construction worker	0.131	0.249	-0.369
X_1	labor-union president			
A	flute player	0.180	0.339	-0.320
X_1	car mechanic			
A	student	0.392	0.439	-0.108
X_1	fashion-monger			
	average	0.220	0.346	-0.280

The average values of the different reported probabilities are

$$p(AX) = 0.22 , \qquad p(X_1) = 0.692 , \qquad p(X_2) = 0.308 ,$$

$$p(AX_1) = 0.346 , \qquad p(AX_2) = 0.154. \tag{96}$$

One can observe that the interference terms fluctuate around a mean of -0.28, with a standard deviation of ± 0.06, that is

$$\overline{q}(AX) = -0.28 \pm 0.06 . \tag{97}$$

There is a clear evidence of the conjunction fallacy, with the conjunction error (91) being $\varepsilon(AX_1) = 0.126$.

QDT interprets the conjunction effect as due to the uncertainty underlying the decision, which leads to the appearance of the intention interferences. The interference of intentions is caused by the hesitation whether, under the given primary feature (A), the object possesses the secondary feature (X_1) or does not have it (X_2). The term $\overline{q}(AX)$ is negative, reflecting the effect of deciding under uncertainty (according to the uncertainty-aversion principle). Quantitatively, we observe that the amplitude $|\overline{q}(AX)|$ is in agreement with the QDT interference-quarter law.

7.5. Combined Conjunction and Disjunction Effects

The QDT predicts that setups in which the conjunction fallacy occurs should also be accompanied by the disjunction effect. To see this, let us extend slightly the previous decision problem by allowing for two representations of the first intention. Concretely, this means that the intention, related to the decision about the primary characteristic, has two representations: (i) "decide about the object or person having or not the primary considered feature" (A), and (ii) "decide to abstain from deciding about this feature" (B). This frames the problem in the context analyzed in the previous section. The conjunction fallacy

occurs when one considers incompatible characteristics [101, 110], such that the probabilities of deciding of having a conjunction (AX_j) or of not guessing about it (BX_j) are close to each other, so that one can set

$$p(A|X_j) = p(B|X_j) \qquad (\forall j) \,. \tag{98}$$

The theorem on interference alternation (Theorem 1) implies that the interference term for being passive under uncertainty is positive and we have

$$q(BX) = -q(AX) > 0 \,. \tag{99}$$

Now, the probability $p(BX)$ of deciding not to guess under uncertainty is governed by an equation similar to Equation (92). Combining this equation with (99), we obtain

$$p(BX) = p(AX) + 2|q(AX)| \,, \tag{100}$$

which shows that, despite equality (98), the probability of being passive is larger than the probability of acting under uncertainty. This is nothing but a particular case of the disjunction effect.

This example shows that the conjunction fallacy is actually a sufficient condition for the occurrence of the disjunction effect, both resulting from the existence of interferences between probabilities under uncertainty. The reverse does not hold: the disjunction effect does not necessarily yield the conjunction fallacy, because the latter requires not only the existence of interferences, but also that their amplitude should be sufficiently large according to the conjunction-fallacy condition (94).

To our knowledge, experiments or situations when the disjunction and conjunction effects are observed simultaneously have not been investigated. The specific prediction coming from the QDT, that the disjunction effect should be observable as soon as the conjunction effect is present, provides a good test of QDT.

8. Non-commutativity of Decisions

It has been mentioned that subsequent decisions, in general, do not commute with each other and that the non-commutativity is intimately connected with the presence of interferences between intentions. As is demonstrated in the previous sections, the phenomenon of intention interference is a key and general phenomenon at the basis of the disjunction effect and conjunction fallacy. Within QDT, we expect it to be generically present in human decision making. We are now in a position to present a rigorous proof that the phenomenon of intention interference is also a crucial ingredient for understanding the non-commutativity of successive decisions.

8.1. Mathematical Formulation of Non-commutativity

To describe in precise mathematical terms the property of non-commutativity, let us consider the case of two intentions. We denote one intention as A. And let the other intention $X \equiv \bigcup_i X_i$, with $i = 1, 2, 3, \ldots$} be composed of several representations X_i, such that the intention A can be certainly realized under one of the intentions X_i from the family X, that is,

$$\sum_i p(X_i|A) = 1 \,. \tag{101}$$

Assume that the joint probabilities are related to the conditional probabilities in the standard way, such that

$$p(AX_i) \equiv p(A|X_i)p(X_i) , \qquad p(X_iA) \equiv p(X_i|A)p(AX) . \qquad (102)$$

For two intended actions A and X_i the following statement holds, demonstrating the non-commutativity of these intended actions.

Theorem 3: *For two intended actions, A and $X = \bigcup_i X_i$, satisfying conditions (101) and (102), the joint probability $p(AX)$ equals $p(XA)$ if and only if there is no interference terms,*

$$p(AX) = p(XA) \quad \leftrightarrow \quad q(AX) = q(XA) \equiv 0 . \qquad (103)$$

And, reciprocally, the intended actions A and X do not commute if and only if the interference factors are nonzero,

$$p(AX) \neq p(XA) , \qquad q(AX) \not\equiv 0 . \qquad (104)$$

Proof: By the general rules of QDT, we have

$$p(XA) = \sum_i p(X_iA) + q(XA) .$$

Employing equations (97) gives

$$p(XA) = \sum_i p(X_i|A)p(AX) + q(XA) .$$

Using normalization (96) yields

$$p(XA) - p(AX) = q(XA) .$$

Interchanging here the actions A and X results in

$$p(XA) - p(AX) = q(AX) .$$

The latter two equations prove the theorem.

The non-commutativity of subsequent decisions is reminiscent of the non-commutativity of subsequent measurements in quantum mechanics. However, there is a principal difference between these phenomena. In decision theory, the prospect states and the strategic state of mind are the *internal states* of the same decision maker. In contrast, in quantum mechanics, the measurement is accomplished by an observer, or an apparatus, which are *external* to the measured physical system. The analogy would be closer, if one could imagine a physical system that attempts to measure some parts of itself. Since standard quantum mechanical measurements do not proceed like this, the mathematics of the non-commutativity of subsequent decisions in decision theory and of subsequent measurements in quantum theory are quite different.

8.2.   *Meaning of Simultaneous Intended Actions*

As follows from the above theorem, when there are two intentions, say A and B, the joint probability $p(AB)$ is generally different from $p(BA)$. Two intentions do not commute with each other, when at least one of them is composite, consisting of several interfering representations, or modes. The intentions commute, only when there is no mode interference. For example, when the mind is one-dimensional or if there is no uncertainty.

Since the order of intended actions is important, when writing $p(AB)$, one has to keep in mind that the intention B is to be realized earlier than A. Even when talking about simultaneous intentions, it is implied that the order AB means the possible realization of B infinitesimally earlier than that of A. To be more precise, let us mark the intention A, associated with time t, as A_t. Respectively, B_t is the intention B, associated with time t. Then the joint probability of these two intentions, taken in the order $A_t B_t$, is defined as

$$p(A_t B_t) \equiv \lim_{t' \to t+0} p(A_{t'} B_t) . \tag{105}$$

Because of the non-commutativity of two intentions, the corresponding decisions also do not commute. Two subsequent decisions, even taken immediately one after another, and under the same circumstances, in general, may lead to different outcomes just as a result of the order of their realization.

9.   Entropy and Information Functional

Quantum decision theory is developed above as a self-consistent mathematical theory. But it remains to be shown how this general theory could be reduced to classical decision theory as a particular case. It is thus necessary to explain how QDT is connected to classical decision theory based on the notion of expected utility. For this purpose, we need to spell out the relation between the utility factor (17) and the classical expected utility. This can be done by invoking conditional entropy maximization, which is equivalent to the minimization of an information functional. The method of conditional entropy maximization is widely used in statistical physics yielding Gibbs ensembles [122]. The method of information minimization is in the basis of the approach to constructing *representative ensembles* [123, 124], using which it is possible to obtain self-consistent description of all, even quite complex, phenomena of statistical physics.

The utility factors $p_0(\pi_j)$ in QDT play the role of classical probabilities. Then the entropy can be defined through these utility factors in the standard way as for any probabilities:

$$S = -\sum_j p_0(\pi_j) \ln p_0(\pi_j) , \tag{106}$$

where the summation is over all prospects π_j pertaining to the given prospect lattice $\mathcal{L}$ (see Definition 5). The normalization condition (18) is valid.

In classical decision theory, one deals with expected utilities defined for the related lotteries [61]. For each prospect π_j, it is possible [78] to put into correspondence a lottery L_j. Therefore, the expected utility for a lottery L_j can be denoted as depending on the prospect π_j corresponding to this lottery. So,

we can write $U(\pi_j)$ for an expected utility of a prospect π_j related to the lottery L_j. For concreteness, we assume that the expected utility is defined so that it is non-negative:

$$U(\pi_j) \geq 0 \qquad (\pi_j \in \mathcal{L}) , \tag{107}$$

which is always possible to achieve.

In addition to the normalization condition (18), we have to impose another condition related to the choice of expected utilities. To this end, we shall use the notion of the likelihood ratio, known in testing statistical theories [125]. In classical decision theory, that lottery is classified as optimal, which provides the maximal expected utility. Treating the expected utility as a likelihood function, we can introduce the likelihood ratio

$$\Lambda(\pi_j) \equiv -\ln \frac{U(\pi_j)}{\sup_j U(\pi_j)} . \tag{108}$$

This likelihood ratio is non-negative, having minimum at zero. The expected likelihood is given by

$$\Lambda \equiv \sum_j p_0(\pi_j)\Lambda(\pi_j) . \tag{109}$$

With entropy (106), under condition (18) and relation (109), the information functional is given by the expression

$$I[p_0(\pi)] = \sum_j p_0(\pi_j) \ln p_0(\pi_j) + \lambda \left[\sum_j p_0(\pi_j) - 1 \right] + \mu \left[\sum_j p_0(\pi_j)\Lambda(\pi_j) - \Lambda \right] , \tag{110}$$

where λ and μ are Lagrange multipliers. This functional is minimized with respect to $p_0(\pi_j)$, when

$$\frac{\delta I[p_0(\pi)]}{\delta p_0(\pi_j)} = 0 , \qquad \frac{\delta^2 I[p_0(\pi)]}{\delta p_0(\pi_j)^2} > 0 . \tag{111}$$

The corresponding variation derivatives yield

$$\frac{\delta I[p_0(\pi)]}{\delta p_0(\pi_j)} = \ln p_0(\pi_j) + 1 + \lambda + \mu\Lambda(\pi_j) ,$$

$$\frac{\delta^2 I[p_0(\pi)]}{\delta p_0(\pi_j)^2} = \frac{1}{p_0(\pi_j)} > 0 . \tag{112}$$

From the first of equations (111), using (108), and denoting

$$Z \equiv e^{1+\lambda} \left[\sup_j U(\pi_j) \right]^{\mu} , \tag{113}$$

which, under normalization (18), becomes

$$Z = \sum_j [U(\pi_j)]^{\mu} , \tag{114}$$

we obtain the utility factor

$$p_0(\pi_j) = \frac{1}{Z} [U(\pi_j)]^{\mu} . \tag{115}$$

Note that, as long as $\mu > 0$, $[U(\pi_j)]^\mu$ also defines a utility function which results in the same ordering as the initial function $U(\pi_j)$. Hence, expression (115) gives an explicit relation between the utility factor in QDT and the expected utility of classical decision theory. From this relation, we see that QDT reduces to classical decision theory when the interference terms vanish so that $p_0(\pi_j) = p(\pi_j)$. The condition $\mu = 1$ in expression (115) is not necessary since $[U(\pi_j)]^\mu$ and $U(\pi_j)$ are two utility functions that result in the same preference ordering. It is natural that the specific value of the Lagrange multiplier μ should be irrelevant in the correspondence between QDT to classical utility theory since μ just quantifies the degree to which the likelihood can vary around some a priori expected likelihood.

10. Conclusions

We have presented a quantum theory of decision making. By its nature, it can, of course, be realized by a quantum object, say, by a quantum computer or another quantum system. This theory provides a guide for creating *thinking quantum systems* [77]. It can be used as a scheme for quantum information processing and for creating artificial intelligence based on quantum laws. This, however, is not compulsory. And the developed theory can also be applied to non-quantum objects with an equal success. It just turns out that the language of quantum theory is a very convenient tool for describing the process of decision making performed by any decision maker, whether quantum or not. In this language, it is straightforward to characterize entangled decisions, non-commutativity of subsequent decisions, and intention interference. These features, although being quantum in their description, at the same time, have natural and transparent interpretations in the simple everyday language and are applicable to the events of real life. To stress the applicability of the approach to the decision making of human beings, we have provided a number of simple illustrative examples.

We have demonstrated the applicability of the approach to the cases when the Savage sure-thing principle is violated, resulting in the disjunction effect. Interference of intentions, arising in decision making under uncertainty, possesses specific features caused by aversion to uncertainty. The theorem on interference alternation that we have derived connects the aversion to uncertainty to the appearance of negative interference terms suppressing the probability of actions. At the same time, the probability of the decision maker not to act is enhanced by positive interference terms. This alternating nature of the intention interference under uncertainty explains the occurrence of the disjunction effect.

The theory has led naturally to a calculational method of the interference terms, based on considerations using robust assessment of probabilities, which makes it possible to predict their influence in a *quantitative* way. The estimates are in good agreement with experimental data for the disjunction effect.

The conjunction fallacy is also explained by the presence of the interference terms. A series of experiments are analyzed and shown to be in excellent agreement with the a priori evaluation of interference effects. The conjunction fallacy is also shown to be a sufficient condition for the disjunction effect, and novel experiments testing the combined interplay between the two effects are suggested.

We have emphasized that the intention interference results in the non-commutativity of subsequent decisions, which follows from the theorem on non-commutativity of intended actions.

The approach of entropy maximization, or information-functional minimization, is employed for deriving a relation between the quantum and classical decision theories.

The specific features of the Quantum Decision Theory, distinguishing it from other approaches known in the literature on decision making and information processing, can be summarized as follows.

(1) QDT is a general mathematical approach that is applicable to arbitrary situations. We do not try to adjust the QDT to fit particular cases; the same theory is used throughout the paper to treat quite different effects.

(2) Each decision maker is characterized by its own strategic state. This strategic state of mind is, generally, not a trivial wave function, but rather a composite vector, incorporating a great number of intended competing actions.

(3) QDT allows us to characterize not a single unusual, quantum-like, property of the decision making process, but several of these characteristics, including entangled decisions, non-commutative decisions, and the interference between intentions.

(4) The literature emphasizes that aversion with respect to uncertainty is an important feeling regulating decision making. This general and ubiquitous feeling is formulated under the uncertainty-aversion principle, connecting it to the signs of the alternating interference terms.

(5) The theorem on interference alternation is proved, which shows that the interference between several intentions, arising under uncertainty, consists of several terms alternating in sign, some being positive and some being negative. These terms are the source of the different paradoxes and logical fallacies presented by humans making decisions in uncertain contexts.

(6) Uncertainty aversion and interference alternation, combined together, are the key factors that suppress the probability of acting and, at the same time, enhance the probability of remaining passive, in the case of uncertainty.

(7) The principal point is that it is not simply the interference between intentions as such, but specifically the interference alternation, together with the uncertainty aversion, which are responsible for the violation of the Savage's sure-thing principle at the origin of the disjunction effect.

(8) The conjunction fallacy is another effect that is caused by the interference of intentions, together with the uncertainty-aversion principle. Without the latter, the conjunction effect cannot be explained.

(9) The conjunction fallacy is shown to be a sufficient condition for the disjunction effect to occur, exhibiting a deep link between the two effects.

(10) The general "interference-quarter law" is formulated, which provides a quantitative prediction for the amplitude of the interference terms, and thus of the quantitative level by which the sure-thing principle is violated.

(11) Detailed quantitative comparisons with experiments documenting the disjunction effect and the conjunction fallacy confirm the validity of the derived laws.

(12) Subsequent decisions are shown, in general, to be not commutative with each other, by proving a theorem on non-commutativity of decisions.

(13) The minimization of an information functional, which is equivalent to the conditional maximization of entropy, makes it possible to connect the quantum probability with expected utility.

(14) The relation between the quantum and classical decision theories is established, showing that the latter is the limit of the former under vanishing interference terms.

Acknowledgements

References

1. Allais, M. Le comportement de l'homme rationnel devant le risque: critique des postulats et axiomes de l'ecole Américaine. *Econometrica* **1953**, *21*, 503-546.
2. *Advances in Behavioral Economics*; Camerer, C. F., Loewenstein, G., Rabin, R., Eds.; Princeton University: Princeton, NI, USA, 2003.
3. Simon, H.A. A behavioral model of rational choice. *Quart. J. Econ.* **1955**, *69*, 99-118.
4. Machina, M.J. Non-expected utility theory. In *The New Palgrave Dictionary of Economics*; Durlauf, S.N., Blume, L.E., Eds.; Macmillan: Basingstoke, UK, 2008.
5. Quiggin, J. A theory of anticiped utility. *J. Econ. Behav. Org.* **1982**, *3*, 323-343.
6. Gilboa, I. Expected utility with purely subjective non-additive probabilities. *J. Math. Econ.* **1987**, *16*, 65-88.
7. Schmeidler, D. Subjective probability and expected utility without additivity. *Econometrica* **1989**, *57*, 571-587.
8. Gilboa, I.; Schmeidler, D. Maxmin expected utility with non-unique prior. *J. Math. Econ.* **1989**, *18*, 141-153.
9. Cohen, M.; Tallon, J.M. Décision dans le risque et l'incertain: l'apport des modèles non additifs. *Rev. Econ. Polit.* **2000**, *110*, 631-681.
10. Montesano, A. Effects of uncertainty aversion on the call option market. *Theory Decis.* **2008**, *65*, 97-123.
11. Dieudonné, J. *Foundations of Modern Analysis*; Hesperides: London, UK, 2006.
12. Bruza, P.; Busemeyer, J.R.; Gaborac, L.; Special issue on quantum cognition. *J. Math. Psychol.* **2009**, *53*, 303-451.
13. Yukalov, V.I.; Sornette, D. Quantum decision theory as quantum theory of measurement. *Phys. Lett. A* **2008**, *372*, 6867-6871.
14. Neumann, von J. *Mathematical Foundations of Quantum Mechanics*; Princeton University: Princeton, NJ, USA, 1955.
15. Yukalov, V.I. Causality problem in quantum physics. *Philos. Sci.* **1975**, *18*, 145-147.
16. Hopfield, J.J. Neural networks and physical systems with emergent collective computational abilities. *Proc. Nat. Acad. Sci. USA* **1982**, *79*, 2554-2558.
17. Hopfield, J.J. Brain, neural networks, and computation. *Rev. Mod. Phys.* **1999**, *71*, 431-437.
18. Amit, D. *Modeling Brain Function*; Cambridge University: Cambridge, UK, 1989.

19. Haken, H. *Brain Dynamics*; Springer: Berlin, Germany, 2008.

20. Penrose, R. *The Emperor's New Mind*; Oxford University: Oxford, UK, 1989.

21. Lockwood, M. *Mind, Brain and the Quantum*; Basil Blackwell: Oxford, UK, 1989.

22. Satinover, J. *The Quantum Brain*; Wiley: New York, NY, USA, 2001.

23. Frölich, H. Bose condensation of strongly excited longitudinal electric modes. *Phys. Lett. A* **1968**, *26*, 402-403.

24. Stuart, C.I.J.; Takahashi, Y.; Umezawa, H. On the stability and non-local properties of memory. *J. Theor. Biol.* **1978**, *71*, 605-618.

25. Stuart, C.I.J.; Takahashi, Y.; Umezawa, H. Mixed system brain dynamics: neural memory as a macroscopic ordered state. *Found. Phys.* **1979**, *9*, 301-327.

26. Beck, F.; Eccles, J. Quantum aspects of brain activity and the role of consciousness. *Proc. Nat. Acad. Sci. USA* **1992**, *89*, 11357-11361.

27. Vitiello, G. Dissipation and memory capacity in the quantum brain model. *Int. J. Mod. Phys. B* **1995**, *9*, 973-989.

28. Hagan, S.; Hameroff, S.R.; Tuszynski, J.A. Quantum computation in brain microtubules: decoherence and biological feasibility. *Phys. Rev. E* **2002**, *65*, 061901.

29. Pessa, E.; Vitiello, G. Quantum noise, entanglement and chaos in the quantum field theory of mind-brain states. *Mind Matter* **2003**, *1*, 59-79.

30. Tegmark, M. Importance of quantum decoherence in brain processes. *Phys. Rev. E* **2000**, *61*, 4194-4205.

31. Chalmers, D. *The Conscious Mind*; Oxford University: Oxford, UK, 1996.

32. Atmanspacher, H.; Römer, H.; Walach, H. Weak quantum theory: complementarity and entanglement in physics and beyond. *Found. Phys.* **2002**, *22*, 379-406.

33. Primas, H. Time-entanglement between mind and matter. *Mind Matter* **2003**, *1*, 81-119.

34. Atmanspacher, H. Mind and matter as asymptotically disjoint inequivalent representations with broken time-reversal symmetry. *Biosystems* **2003**, *68*, 19-30.

35. Stapp, H.P. *Mind, Matter, and Quantum Mechanics*; Springer: Berlin, Germany, 1993.

36. Stapp, H.P. Attention, intention, and will in quantum physics. *J. Consciousn. Stud.* **1999**, *6*, 143-164.

37. Bohr, N. Wirkungsquantum und Naturbeschreibung. *Naturwissenschaft* **1929**, *17*, 483-486.

38. Bohr, N. Light and life. *Nature* **1933**, *131*, 421-423, 457-459.

39. Bohr, N. Kausalität und Komplemetarität. *Erkenntnissenscahft* **1937**, *6*, 293-303.

40. Bohr, N. *La Physique Atomique et la Connaissance Humaine*; Gontier: Geneva, Switzerland, 1961.

41. Meyer, D. Quantum strategies. *Phys. Rev. Lett.* **1999**, *82*, 1052-1055.

42. Goldenberg, L.; Vaidman, L.; Wiesner, S. Quantum gambling. *Phys. Rev. Lett.* **1999**, *82*, 3356-3359.

43. Eisert, J.; Wilkens, M. Quantum games. *J. Mod. Opt.* **2000**, *47*, 2543-2556.

44. Johnson, N.F. Playing a quantum game with a corrupted source. *Phys. Rev. A* **2001**, *63*, 020302.

45. Benjamin, S.C.; Hayden, P.M. Multi-player quantum games. *Phys. Rev. A* **2001**, *64*, 030301.

46. Iqbal, A.; Toor, A.H. Evolutionally stable strategies in quantum games. *Phys. Lett. A* **2001**, *280*, 249-256.

47. Li, C.F.; Zhang, Y.S.; Huang, Y.F.; Guo, G.C. Quantum strategies of quantum measurements. *Phys. Lett. A* **2001**, *280*, 257-260.

48. Du, J.; Xu, X.; Li, H.; Zhou, X.; Han, R. Entanglement playing a dominating role in quantum games. *Phys. Lett. A* **2001**, *289*, 9-15.

49. Du, J.; Li, H.; Xu, X.; Shi, M.; Wu, J.; Zhou, X.; Han, R. Experimental realization of quantum games on a quantum computer. *Phys. Rev. Lett.* **2002**, *88*, 137902.

50. Lee, C.F.; Johnson, N.F. Efficiency and formalism of quantum games. *Phys. Rev. A* **2003**, *67*, 022311.

51. Enk, van S.J.; Pike, R. Classical rules in quantum games. *Phys. Rev. A* **2002**, *66*, 024306.

52. Landsburg, S.E. Quantum game theory. *Am. Math. Soc.* **2004**, *51*, 394-399 .

53. Cheon, T.; Tsutsui, I. Classical and quantum contents of solvable game theory on Hilbert space. *Phys. Lett. A* **2006**, *348*, 147-152.

54. Guo, H.; Zhang, J.; Koehler, G.J. A survey of quantum games. *Decis. Support Syst.* **2008**, *46*, 318-332.

55. Shor, P. Polynomial-type algorithms for prime factorization and discrete logarithms on a quantum computer. *SIAM J. Sci. Stat. Comput.* **1997**, *26*, 1484-1494.

56. Nottale, L. Scale relativity and quantization of extra-solar planetary systems. *Astron. Astrophys. Lett.* **1996**, *315*, L9-L12.

57. Nottale, L.; Schumacher, G.; Gay, J. Scale relativity and quantization of the solar system. *Astron. Astrophys.* **1997**, *322*, 1018-1025.

58. Nottale, L.; Schumacher, G.; Lefèvre, E.T. Scale relativity and quantization of exoplanet orbital semi-major axes. *Astron. Astrophys.* **2000**, *361*, 379-387.

59. De Broglie, L. *Les Incertitudes d'Heisenberg et l'Interpretation Probabiliste de la Mecanique Ondulatoire*; Gauthier-Villars: Paris, France, 1982.

60. Bohm, D. *Quantum Theory*; Prentice-Hall: London, UK, 1951.

61. Neumann, von J.; Morgenstern, O. *Theory of Games and Economic Behavior*; Princeton University: Princeton, NJ, USA, 1944.

62. Savage, L.J. *The Foundations of Statistics*; Wiley: New York, NY, USA, 1954.

63. Lindgren, B.W. *Elements of Decision Theory*; Macmillan: New York, NY, USA, 1971.

64. White, D.I. *Fundamentals of Decision Theory*; Elsevier: New York, NY, USA, 1976.

65. Hastings, N.A.; Mello, J.M. *Decision Networks*; Wiley: Chichester, UK, 1978.

66. Rivett, P. *Model Building for Decision Analysis*; Wiley: Chichester, UK, 1980.

67. Buchanan, J.T. *Discrete and Dynamic Decision Analysis*; Wiley: Chichester, UK, 1982.

68. Berger, J.O. *Statistical Decision Theory and Bayesian Analysis*; Springer: New York, NY, USA, 1985.

69. Marshall, K.T.; Oliver, R.M. *Decision Making and Forecasting*; McGraw-Hill: New York, NY, USA, 1995.

70. Bather, J. *Decision Theory*; Wiley: Chichester, UK, 2000.

71. French, S.; Insua, D.R. *Statistical Decision Theory*; Arnold: London, UK, 2000.

72. Raiffa, H.; Schlaifer, R. *Applied Statistical Decision Theory*; Wiley: New York, NY, USA, 2000.

73. Weirich, P. *Decision Space*; Cambridge University: Cambridge, UK, 2001.

74. Weidlich, W. Physics and social science–the approach of synergetics. *Phys. Rep.* **1991**, *204*, 1-163.

75. Zeckhauser, R. Investing in the unknown and unknowable. *Capitalism Soc.* **2006**, *1*, 1-39.

76. Benioff, P.A. Decision procedures in quantum mechanics. *J. Math. Phys.* **1972**, *13*, 909-915.

77. Yukalov, V.I.; Sornette, D. Scheme of thinking quantum systems. *Laser Phys. Lett.* **2009**, *6*, 833-839.

78. Yukalov, V.I.; Sornette, D. Physics of risk and uncertainty in quantum decision making. *Eur. Phys. J. B* **2009**, *71*, 533-548.

79. Dirac, P.A.M. *The Principles of Quantum Mechanics*; Clarendon: Oxford, UK, 1958.

80. Yukalov, V.I. Entanglement measure for composite systems. *Phys. Rev. Lett.* **2003**, *90*, 167905.

81. Yukalov, V.I. Quantifying entanglement production of quantum operations. *Phys. Rev. A* **2003**, *68*, 022109.

82. Einstein A.; Podolsky B.; Rosen N. Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.* **1935**, *47*, 777-780.

83. Bell, J.S. On the Einstein-Podolsky-Rosen paradox. *Physics* **1964**, *1*, 195-200.

84. Yukalov, V.I. Evolutional entanglement in nonequilibrium processes. *Mod. Phys. Lett. B* **2003**, *17*, 95-103.

85. Bechara, A.; Damasio, H.; Damasio, A. Emotion, decision making and the orbitofrontal cortex. *Cerebral Cortex* **2000**, *10*, 295-307.

86. Dickhaut, J.; McCabe, K.; Nagode, J.C.; Rustichini, A.; Smith, K.; Pardo, J.V. The impact of the certainty context on the process of choice. *Proc. Nat. Acad. Sci. USA* **2003**, *100*, 3536-3541.

87. Dawkins, R. *The Selfish Gene*; Oxford University: New York, NY, USA, 2006.

88. Tversky, A.; Kahneman, D. Availability: a heuristic for judging frequency and probability. *Cognit. Psychol.* **1973**, *5*, 207-232.

89. Kaplan, S.; Garrick, B.J. On the quantitative definition of risk. *Risk Anal.* **1981**, *1*, 11-27.

90. Frederick, S.; Loewenstein, G.; O'Donoghue, T. Time discounting and time preference: a critical review. *J. Econ. Lit.* **2002**, *40*, 351-401.

91. Rambaud, S.C.; Torrecillas, M.J. Some considerations on the social discount rate. *Environm. Sci. Policy* **2005**, *8*, 3453-355.

92. Barkan, R.; Danziger, S.; Ben-Bashat, G.; Busemeyer, J.R. Framing reference points: the effect of integration and segregation on dynamic inconsistency. *J. Behav. Decis. Making* **2005**, *18*, 213-226.

93. Busemeyer, J.R.; Wang, Z.; Townsend, J.T. Quantum dynamics of human decision-making. *J. Math. Psychol.* **2006**, *50*, 220-241.

94. Browning, R. *Complete Poetical Works*; Houghton and Miffin: Cambridge, UK, 1895.

95. Feller, W. *Introduction to Probability Theory and Its Applications*; Wiley: New York, NY, USA, 1970.

96. Tversky, A.; Shafir, E. The disjunction effect in choice under uncertainty. *Psychol. Sci.* **1992**, *3*, 305-309.

97. Croson, R.T.A. The disjunction effect and reason-based choice in games. *Org. Behav. Human Decis. Processes* **1999**, *80*, 118-133.

98. Lambdin, C.; Burdsal, C. The disjunction effect reexamined: relevant methodological issues and the fallacy of unspecified percentage comparisons. *Org. Behav. Human Decis. Processes* **2007**, *103*, 268-276.

99. Li, S.; Taplin, J.E.; Zhang, Y. The equate-to-differentiate way of seeing the prisoner's dilemma. *Inform. Sci.* **2007**, *177*, 1395-1412.

100. Shafir, E.; Tversky, A. Thinking through uncertainty: nonconsequential reasoning and choice. *Cognit. Psychol.* **1992**, *24*, 449-474.

101. Shafir, E.B.; Smith, E.E.; Osherson, D.N. Typicality and reasoning fallacies. *Memory Cognition* **1990**, *18*, 229-239.

102. Shafir, E. Uncertainty and the difficulty of thinking through disjunctions. *Cognition* **1994**, *50*, 403-430.

103. Kühberger, A.; Komunska D.; Perner, J. The disjunction effect: does it exist for two-step gambles? *Org. Behav. Human Decis. Processes* **2001**, *85*, 250-264.

104. Shafir, E.; Simonson, I.; Tversky, A. Reason-based choice. *Cognition* **1993**, *49*, 11-36.

105. Epstein, L.G. A definition of uncertainty aversion. *Rev. Econ. Stud.* **1999**, *66*, 579-608.

106. Tversky, A.; Koehler, D. Support theory: a nonexistential representation of subjective probability. *Psychol. Rev.* **1994**, *101*, 547-567.

107. Fox, C.; Rogers, B.; Tversky, A. Option traders exhibit subadditive decision weights. *J. Risk Uncert.* **1996**, *13*, 5-17.

108. Rottenstreich Y.; Tversky, A. Unpacking, repacking and anchoring: advances in support theory. *Psychol. Rev.* **1997**, *104*, 406-415.

109. Tversky, A.; Kahneman, D. Judgements of and by representativeness. In *Judgements Under Uncertainty: Heuristics and Biases*; Kahneman, D., Slovic, P., Tversky, A. Eds.; Cambridge University: New York, NY, USA, 1980; pp. 84-98.

110. Tversky, A.; Kahneman, D. Extensional versus intuitive reasoning: the conjunction fallacy in probability judgement. *Psychol. Rev.* **1983**, *90*, 293-315.

111. Morier, D.M.; Borgida, E. The conjunction fallacy: a task-specific phenomenon? *Personal. Soc. Psychol. Bull.* **1984**, *10*, 243-253.

112. Wells, G.L. The conjunction error and the representativeness heuristic. *Soc. Cognit.* **1985**, *3*, 266-279.

113. Yates, J.F.; Carlson, B.W. Conjunction errors: evidence for multiple judgement procedures, including signed summation. *Org. Behav. Hum. Decis. Processes* **1986**, *37*, 230-253.

114. Tentori, K.; Bonini, N.; Osherson, D. The conjunction fallacy: a misunderstanding about conjunction? *Cognit. Sci.* **2004**, *28*, 467-477.

115. Sornette, D.; Davis, A.B.; Ide, K.; Vixie, K.R.; Pisarenko, V.; Kamm, J.R. Algorithm for model validation: theory and applications. *Proc. Nat. Acad. Sci. USA* **2007**, *104*, 6562-6567.

116. Kahneman, D.; Tversky, A. Prospect theory: an analysis of decisions under risk. *Econometrica* **1979**, *77*, 263-291.

117. Kahneman, D.; Lovallo, D. Timid choices and bold forecasts: a cognitive perspective on risk-taking. *Manag. Sci.* **1993**, *39*, 17-31.

118. Legrenzi, P.; Girotto, V.; Johnson-Laird, P.N. Focusing in reasoning and decision making. *Cognition* **1993**, *49*, 36-66.

119. Baron, J. *Judgement Misguided: Intuition and Error in Public Decision-Making*; Oxford University: Oxford, UK, 1998.

120. Read, D.; Loewenstein, G.L.; Rabin, M. Choice bracketing. *J. Risk Uncert.* **1999**, *19*, 171-197.

121. Loewenstein, G.; Rick, S.; Cohen, J.D. Neuroeconomics. *Annu. Rev. Psychol.* **2008**, *59*, 647-672.

122. Balescu, R. *Equilibrium and Nonequilibrium Statistical Mechanics*; Wiley: New York, NY, USA, 1975.

123. Yukalov, V.I. Phase transitions and heterophase fluctuations. *Phys. Rep.* **1991**, *208*, 395-492.

124. Yukalov, V.I. Representative ensembles in statistical mechanics. *Int. J. Mod. Phys. B* **2007**, *21*, 69-86.

125. Sornette, D. *Critical Phenomena in Natural Sciences*, 2nd ed.; Springer: Berlin, Gemany, 2006.