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Article

Information, Deformed κ-Wehrl Entropies and Semiclassical Delocalization

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Abstract: Semiclassical delocalization in phase space constitutes a manifestation of the Uncertainty Principle, one indispensable part of the present understanding of Nature and the Wehrl entropy is widely regarded as the foremost localization-indicator. We readdress the matter here within the framework of the celebrated semiclassical Husimi distributions and their associated Wehrl entropies, suitably $\kappa-$deformed. We are able to show that it is possible to significantly improve on the extant phase-space classical-localization power.

Keywords: Uncertainty principle; classical localization; escort distributions; semiclassical methods.

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1. Introduction

The uncertainty principle has, of course, manifestations in phase space within semi-classical strictures. The celebrated Wehrl W entropy reflects indeterminacy via the so-called Lieb bound $W \geq 1$ [1, 2]. We revisit the subject here by recourse to a novel (within the field) conjunction of two important statistical tools: i) escort distributions (EDs) of a given order [3], and ii) $\kappa-$deformed entropies (see [4] and references therein).

The ED is a rather well established tool in mathematical circles although it remains (for physicists) a somewhat novel concept (rapidly gaining acceptance in the physics world, though). The semi-classical techniques that we are going to employ for applying/relating i) the concomitant "escort distribution" concepts in conjunction with ii) information measures (expressed in *phase-space* parlance) are discussed at length in, for example, in Ref. [5].

It is well-knows that the oldest and most elaborate phase-space formulation of quantum mechanics is that of Wigner [6–8] that assigns to every quantum state a PS-function (the Wigner one). This function can assume negative values so that instead of being a probability distribution it is regarded as a quasi-probability density. The negative-values' feature was circumvented by Husimi [9] (among others), in terms of probability distributions $\mu(x,p)$ today baptized as the Husimi ones [10]. The whole of quantum mechanics can be completely reformulated in $\mu(x,p)-$terms [11, 12]. This distribution μ can be viewed as a "smoothed Wigner distribution" [7]. In point of fact, $\mu(x,p)$ is a Wigner–distribution D_W, smeared over an $\hbar$ sized region (cell) of phase-space [13]. Such smearing renders $\mu(x,p)$ a positive function, even if D_W does not have such a character. The semi-classical $\mu-$PD alludes to a *particular kind* of PD: that for simultaneous, even if approximate, location of position and momentum in phase space [13].

As for the present endeavor one is to underline that, as we will show below, *methods for localization in phase space* can be improved by suitably

- choosing an escort order q and
- "deforming" the Wehrl entropy (see Sect. 4) up to a degree, say, κ.

To such an end, surprisingly enough, one needs to bring into the picture a purely quantum notion, namely, that of "participation ratio of a mixed state". This entails that purely quantum concepts already appear at the semiclassical level.

Our motivation here resides in the fact that understanding the emergence of classical behavior is one of the great problems of contemporary physics [14]. Additionally, the subject of phase space localization is of importance in the fascinating field of Quantum Chaos (see, for example, [15–18] and references therein). Our work is organized as follows: Section 2 introduces the prerequisites, while our new results are given in Sects. 3-4. Finally, some conclusions are drawn in Sect. 5.

2. Preliminary materials

2.1. Escort distributions

Consider two (normalized) probability distributions $f(x)$, $f_q(x)$, and an "operator" $\hat{O}^q$ linking them in the fashion

$$f_q(x) = \hat{O}^q \; f(x) = \frac{f(x)^q}{\int \mathrm{d}x \, f(x)^q}. \tag{1}$$

We say that $f_q(x)$ is the order $q-$associated escort distribution of f, with $q \in \Re$. Often, f_q is often able to discern in better fashion than f important details of the phenomenon at hand [3, 5].

The expectation value of a quantity $\mathcal{A}$ evaluated with a $q-$escort distribution will be denoted by $\langle \mathcal{A} \rangle_{f_q}$. For some physical applications of the concept in statistical mechanics see, for instance (not an exhaustive list), [5, 19–21], and references therein. For physicists, the fundamental reference on escort distributions is [3].

2.2. Semi-classical Husimi distributions, Wehrl entropy, and Fisher information

Here we wish to introduce the ED-tool into a scenario involving semiclassical Husimi distributions and thereby try to gather *semiclassical information* from escort Husimi distributions ($q-$HDs). Some ideas that we abundantly use below can be found in Ref. [21]. In this work we also discuss an important information instrument expressed in *phase-space vocabulary*, namely, the semi-classical Wehrl entropy W, a useful *measure of localization in phase-space* [1, 22]. It is built up using coherent states $|z\rangle$ [13, 23] and constitutes a powerful tool in statistical physics. Of course, coherent states are eigenstates of a general annihilation operator $\hat{a}$, appropriate for the problem at hand [23–25], i.e.,

$$\hat{a}|z\rangle = z|z\rangle, \tag{2}$$

with z a complex combination of the phase-space coordinates x, p ($\hat{a}$ is not Hermitian),

$$z = (m\omega/2\hbar)^{1/2}x + i(\hbar\omega m/2)^{-1/2}p \tag{3}$$

The pertinent $W-$definition reads

$$W = -\int \mathrm{d}\Omega\, \mu(x,p)\, \ln \mu(x,p), \quad \mathrm{d}\Omega = \frac{\mathrm{d}x\mathrm{d}p}{2\pi\hbar} = \frac{\mathrm{d}^2 z}{\pi} \tag{4}$$

clearly a Shannon-like measure [26] to which MaxEnt considerations can be applied. W is expressed in terms of distribution functions $\mu(x,p)$, *the leitmotif of the present work*, commonly referred to as Husimi distributions [9]. As an important measure of localization in phase-space, W possesses a lower bound, related to the Uncertainty Principle and demonstrated by Lieb with reference to the harmonic oscillator coherent states [2]

$$W \geq 1, \tag{5}$$

on which we wish here to improve upon in the present communication.

Husimi's μ's are the diagonal elements of the density operator $\hat{\rho}$, that yields all the available physical information concerning the system at hand [27], in the coherent state basis $\{|z(x,p)\rangle\}$ [23], i.e.,

$$\mu(x,p) \equiv \mu(z) = \langle z|\hat{\rho}|z\rangle. \tag{6}$$

Thus, they are "semi-classical" phase-space distribution functions associated to the system's $\hat{\rho}$ [23–25]. The distribution $\mu(x,p)$ is normalized in the fashion

$$\int \mathrm{d}\Omega\, \mu(x,p) = 1. \tag{7}$$

It is shown in Ref. [28] that, in the all-important harmonic oscillator (HO) instance, the associated Husimi distribution reads

$$\mu(x,p) \equiv \mu(z) = (1 - e^{-\beta\hbar\omega})\, e^{-(1-e^{-\beta\hbar\omega})|z|^2}, \tag{8}$$

with $\beta = 1/k_B T$, T the temperature, which leads to a pure Gaussian form in the $T = 0-$limit.

Of course, it is clear that the q-escort Husimi distribution $\gamma_q(x,p)$ will read

$$\gamma_q(x,p) = \hat{O}^q \, \mu(x,p) = \frac{\mu(x,p)^q}{\int \mathrm{d}\Omega \, \mu(x,p)^q}. \tag{9}$$

Escort Husimi distributions attain a larger degree of phase-space localization-power as q grows, but too large $q-$values cannot be accepted because they lead to mathematical absurdities (see the forthcoming Section).

We can also easily construct a "semiclassical" Fisher information measure I_F^{sc} following the above Wehrl-methodology (see, for instance, [28–30] and references therein). The classical Fisher information I_x associated with translations of a one-dimensional observable x with corresponding probability density $\rho(x)$ is [30]

$$I_x = \int \mathrm{d}x \, \rho(x) \left[\frac{\partial \ln \rho(x)}{\partial x} \right]^2, \tag{10}$$

and the Cramer–Rao inequality is given by [30]

$$\Delta x \geq I_x^{-1} \tag{11}$$

where Δx is the variance for the stochastic variable x which is of the form [30]

$$\Delta x^2 = \langle x^2 \rangle - \langle x \rangle^2 = \int \mathrm{d}x \, \rho(x) \, x^2 - \left(\int \mathrm{d}x \, \rho(x) \, x \right)^2. \tag{12}$$

In semiclassical terms, with a Husimi distribution playing the role of $\rho(x)$, one has [28]

$$I_F^{sc} = \frac{1}{4} \int \frac{\mathrm{d}^2 z}{\pi} \left\{ \frac{\partial \ln \mu(z)}{\partial |z|} \right\}^2, \tag{13}$$

so that inserting the HO-Husimi distribution into the above expression we obtain the analytic form [28]

$$I_F^{sc} = 1 - e^{-\beta \hbar \omega}, \tag{14}$$

so that

$$\begin{aligned} &\text{For } T \rightarrow 0 \text{ one has } I_F^{sc} = 1 \\ &\text{For } T \rightarrow \infty \text{ one has } I_F^{sc} = 0, \end{aligned} \tag{15}$$

as it should be expected. Additionally, the Fisher measure as a functional of an escort-Husimi distribution reads [21]

$$I_F^{scq} = q I_F^{sc}; \;\; \Delta x I_F^{scq} \geq q \tag{16}$$

3. Participation ratio

In order to ascertain what $q-$values (order of the escort distribution) make physical sense one needs to revisit a purely quantal concept [5]: that of "degree of purity" of a general density operator (and then of a mixed state), which is expressed via $\mathrm{Tr}\,\hat{\rho}^2$ [31, 32]. Its inverse, the so-called participation ratio is

$$\mathcal{R}(\rho) = \frac{1}{\mathrm{Tr}\,\hat{\rho}^2}, \tag{17}$$

and tells us about the number of pure state-projectors that enter $\hat{\rho}$. Expression (17) is particularly convenient for calculations [32]. For pure states idem-potency reigns $\hat{\rho}^2 = \hat{\rho}$ and thus $\mathcal{R}(\hat{\rho}) = 1$. In the case of mixed states it is always true that

$$\mathcal{R}(\hat{\rho}) \geq 1, \tag{18}$$

because $\hat{\rho}^2 \neq \hat{\rho}$ and $\mathrm{Tr}\,\hat{\rho}^2 \leq 1$. By recourse to the $z-$notation introduced in (6) one can evaluate Eq. (17) (a "semi-classical" participation ratio) as follows:

$$\mathcal{R}(\mu) = \frac{1}{\int \mathrm{d}\Omega \mu(z)^2}, \tag{19}$$

where $\mu(z) = \langle z|\hat{\rho}|z\rangle$ is our Husimi function and the phase-space volume $\mathrm{d}\Omega$ in classical mechanics is related to differential element $\mathrm{d}^2 z$ in the form $\mathrm{d}\Omega = \mathrm{d}^2 z/\pi$ [23]. Notice that, at $T = 0$, the participation ratio of an HO-Husimi distribution [Cf. Eq. (8)] equals two, which is a reflection of its semiclassical nature.

Now we can extend the $\mathcal{R}-$definition to γ_q of (9) and find

$$\gamma_q(z) = \frac{\mu(z)^q}{\int \frac{\mathrm{d}^2 z}{\pi}\, \mu(z)^q}, \tag{20}$$

where $\mu(z)$ stands for the ordinary Husimi distribution. One can compute it easily for the harmonic oscillator by recourse to the result given in [21] for the HO-escort-Husimi distribution μ_{HO}, obtaining [Cf. Eq. (8)] [21]

$$\gamma_q(z) = q\,(1 - e^{-\beta\hbar\omega})^{1-q}\,(\mu(z))^q. \tag{21}$$

Accordingly, in this case the pertinent "semi-classical" participation ratio reads

$$\mathcal{R}_q(\gamma_q) = \frac{1}{\int \frac{\mathrm{d}^2 z}{\pi} \gamma_q(z)^2} = \frac{2}{q(1 - \exp(-\beta\hbar\omega))}. \tag{22}$$

Enforcing Eq. (18), that *necessarily holds* from the definition (22), we reach the inequality

$$\frac{2}{q(1 - \exp(-\beta\hbar\omega))} \geq 1, \tag{23}$$

that sets an upper bound q^* to the escort degree q. For $T \to 0$ we have $q^* = 2$. As T grows, so does q^*. A similar line of reasoning gives, for finite temperatures, a $q-$upper bound that varies with T, as is evident from Eq. (23). As a consequence, for $T \to \infty$,

$$\gamma_q(z) \to q\beta\hbar\omega\, e^{-q\beta\hbar\omega\,|z|^2}, \tag{24}$$

yielding $\mathcal{R}_q \to \infty$, as expected. Let us insist on the fact that, for $T = 0$, we have

$$\mathcal{R}_q(T=0) = \frac{2}{q} \Rightarrow \text{that for } q=2,\ \mathcal{R}_q(T=0)=1, \tag{25}$$

so that the $q-$Husimi distribution becomes that of a pure state. From the preceding considerations we gather that, since $1 \leq \mathcal{R}_q(\gamma_q) \leq \infty$,

$$0 \leq q \leq 2, \tag{26}$$

the main result of this section.

Eq. (26) should be considered as an improvement on Eq. (18) of [5], that sets a $q-$upper bound of $\sqrt{2}$. Summing up, escort Husimi distributions attain a large degree of phase-space localization-power as q increases, but too large $q-$values are unacceptable because they lead us to mathematical absurdities. Finally, one can also think of Eq. (22) as providing us with a sort of new "saturated" Cramer-Rao relationship [29] [Cf. (14)]

$$qI_F^{sc}\mathcal{R}_q(\gamma_q) = 2;\quad I_F^{sc} \in [0,1];\quad q \in [0,2];\quad \mathcal{R}_q(\gamma_q) \in [1,\infty] \tag{27}$$

relating the participation ratio to the semiclassical Fisher information, also a novel result (the second here).

4. Deformed κ-Wehrl measures

Wehrl's entropy for a Husimi distribution is just Shannon's measure

$$W = -\int \frac{\mathrm{d}^2 z}{\pi}\mu(z)\ln[\mu(z)], \tag{28}$$

and satisfies the Lieb bound $W \geq 1$ (equal sign at zero temperature). Instead, the same information measure, but now for an escort-Husimi, $\gamma_q(z)$ verifies a $q-$dependent bound, namely [33],

$$W(\gamma_q) \geq 1 - \ln(q), \tag{29}$$

which, given that the maximum possible value for q is two, as seen above, yields $W(\gamma_q) \geq 1 - \ln(2) \approx 0.31$, a considerable improvement upon Lieb's value.

Following [21] we consider in this Section still another semi-classical entropy, a deformed $\kappa-$Wehrl one, that derives not from Shannon's measure but from Tsallis' one. This is done by recourse to the so-called $\kappa-$logarithm [4]

$$\ln_\kappa(x) = \frac{1 - x^{1-\kappa}}{\kappa - 1}. \tag{30}$$

One has then the "deformed" measure of a Husimi distribution [21]

$$W_\kappa[\mu] = -\int \frac{\mathrm{d}^2 z}{\pi}\mu(z)^\kappa \ln_\kappa[\mu(z)], \tag{31}$$

that, given the HO-Husimi form of $\mu(z)$ yields

$$W_\kappa[\mu] = \frac{1}{\kappa} + \ln_\kappa[1/(1-\exp(-\beta\hbar\omega))]. \tag{32}$$

Tsallis measure is extremized by the so-called $\kappa-$exponentials $e_\kappa(x)$, which are the suitable inverse of the $\kappa-$logarithm (i.e., $\ln_\kappa[e_\kappa(x)] = x = e_\kappa[\ln_\kappa(x)]$). These $\kappa-$exponentials are power law distributions [4], that have finite variance, as maximum entropy extremizers with a mean-energy constraint, for $\kappa \leq 5/3$ [4]. Thus, in such a sense, the maximum permissible $\kappa-$value is 5/3. If we drop the finiteness-requirement, then the maximum allowable $\kappa-$value augments up to 3. For $\kappa > 3$ normalization of the power law distribution ceases to be possible [4]. As it is obvious, the new quantity is bounded in different fashion than the Wehrl measure, i.e.,

$$W_\kappa[\mu] \geq \frac{1}{\kappa}, \tag{33}$$

equality holding for $T = 0$. Since κ can be greater than unity, we are definitely improving upon Lieb's bound by diminishing it from unity to 1/3 (*our third novel result here*). This bound can still be further upgraded by replacing the Husimi distribution $\mu(z)$ by the "more peaked" PDF that arises using the escort distribution concept, like $\gamma_q(z)$. In such an instance one finds,

$$W_\kappa[\gamma_q(z)] = \frac{1}{\kappa} + \frac{1}{\kappa}\ln_\kappa[1/q(1-\exp(-\beta\hbar\omega))], \tag{34}$$

obeying

$$W_\kappa \geq \frac{1}{\kappa}(1 + \ln_\kappa(1/q)). \tag{35}$$

Notice that we are not employing a "more $\kappa-$generalized" HO-distribution but a escort-modified Husimi-HO one. $W_\kappa(\gamma_q)$ must be of course a definite-positive measure, and it is so only in that region of the $\kappa - q-$plane delimited by the inequality $1 + \ln_\kappa(1/q) \geq 0$, as we can see in Fig. 1, that is

$$\kappa q^{1-\kappa} \geq 1 \tag{36}$$

which improves upon the bound (33) since the second term in the r.h.s. of (35) can be negative (*a fourth result*). Also, we see that q and κ are not independent variables.

5. Conclusions

In the present endeavor one we have dealt with methods for localization in phase space, showing that they can be improved by suitably choosing

- the escort order q and
- the degree of deformation, κ, of a Wehrl entropy.

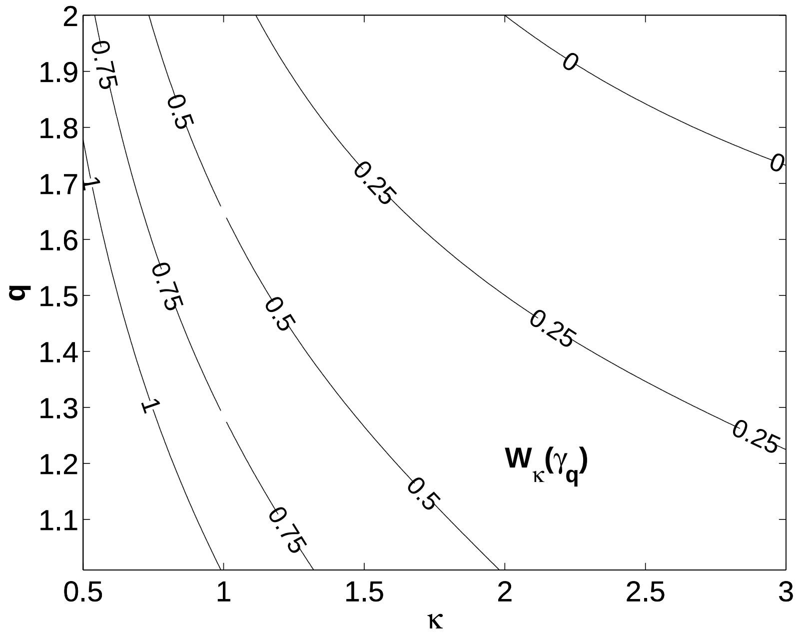
To such an end one needs recourse to the purely quantum notion of participation ratio of a mixed state. Four new results have been obtained.

- We have obtained the inequality

$$\frac{2}{q(1-\exp(-\beta\hbar\omega))} \geq 1, \tag{37}$$

  that sets an upper bound q^* to the possible escort degrees q of a Husimi distribution. For $T \to 0$ one has $q^* = 2$. As T grows, so does q^*.

Figure 1. We display the relationship between our two parameters q and κ. They range within the $q-$ and $\kappa-$upper values here indicated, below the "0-contour-curve".



- One encounters a kind of new "saturated" Cramer-Rao relationship [29] [Cf. (14)]

$$qI_F^{sc}\mathcal{R}_q(\gamma_q) = 2; \;\; I_F^{sc} \in [0,1]; \;\; q \in [0,2]; \;\; \mathcal{R}_q(\gamma_q) \in [1,\infty] \tag{38}$$

  relating the participation ratio to the semiclassical Fisher information.

- For our deformed Wehrl functions we find

$$W_\kappa[\mu] \geq \frac{1}{\kappa}, \tag{39}$$

  equality holding for $T = 0$. Since κ can be greater than unity, we are definitely improving upon the celebrated Lieb's bound $W \geq 1$.

- These deformed functions $W_\kappa(\gamma_q) \geq 0$, which is true only in that region of the $\kappa - q-$plane delimited by the inequality

$$\kappa q^{1-\kappa} \geq 1, \tag{40}$$

  which, according to (35), improves upon the bound (39). Notice that, as a consequence, our two parameters q and κ are not free, but are linked by the above inequality.

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