

Immersing the Schwarzschild black hole in test nonlinear electromagnetic fields

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- Wald's solution - rotating black hole immersed in a uniform magnetic field
- generalisation of Wald's solution: we take into account the effects of **nonlinear electrodynamics (NLE)**
- NLE Lagrangian density - a smooth function of two electromagnetic invariants

Astrophysical observations

- EM fields surround astrophysical black holes
 - needed to explain jets of matter coming from active galactic nuclei?
 - magnetars possess extreme magnetic fields (up to 10^{11}T) - NLE effects could be noticed

Theoretical research

- EM potentials and charges appear in laws of black hole thermodynamics
- NLE models might "regularise" black hole singularities

Killing vectors as gauge fields

The source-free Maxwell's equations:

$$dF = 0$$

$$d * F = 0$$

ansatz: $F = dK$, where K^a is a Killing vector field

- by definition $dF = 0$
- the Killing lemma: $\nabla_b \nabla^b K^a = -R^a_c K^c$
- in vacuum spacetimes: $R_{ab} = 0$, so $d * F = 0$

→ Killing vectors satisfy the source-free Maxwell's equations

→ F is a test EM field

Wald's solution

Kerr black hole in an asymptotically homogeneous magnetic field

- $k = \partial/\partial t$ stationary Killing vector
- $m = \partial/\partial\phi$ axial Killing vector
- B_∞ field strength at infinity
- $a = J/M$ ratio of black hole's angular momentum J and mass M

Test electromagnetic field 2-form:

$$F = \frac{1}{2} B_\infty (2adk + dm)$$

Electric charge: $Q_\infty = \frac{1}{4\pi} \oint_{\mathcal{S}_\infty} *F = B_\infty (-2aM + 2J) = 0$ ⁽¹⁾

Magnetic charge: $P_\infty = \frac{1}{4\pi} \oint_{\mathcal{S}_\infty} F = 0$

⁽¹⁾ \mathcal{S}_∞ denotes sphere at infinity

Generalised Maxwell's equations:

$$dF = 0$$

$$d * Z = 0, \text{ with } Z = -4(\mathcal{L}_{\mathcal{F}}F + \mathcal{L}_{\mathcal{G}} \star F)$$

Notation: \mathcal{L}_x denotes $\partial_x \mathcal{L}$

Electromagnetic invariants: $\mathcal{F} = F_{ab}F^{ab}$, $\mathcal{G} = F_{ab} \star F^{ab}$

Notable NLE Lagrangians:

- Euler-Heisenberg theory - 1-loop QED correction

$$\mathcal{L}^{(\text{EH})} = -\frac{1}{4}\mathcal{F} + \frac{\alpha^2}{360m_e^4} (4\mathcal{F}^2 + 7\mathcal{G}^2)$$

- Born-Infeld theory - phenomenological

$$\mathcal{L}^{(\text{BI})} = b^2 \left(1 - \sqrt{1 + \frac{\mathcal{F}}{2b^2} - \frac{\mathcal{G}^2}{16b^4}} \right) = -\frac{1}{4}\mathcal{F} + \frac{1}{32b^2}(\mathcal{F}^2 + \mathcal{G}^2) + \dots$$

Generalisation of Wald's solution to NLE

We are looking for an exact solution of NLE Maxwell's equations:

1) using the basic ansatz $F = dK$

- $d \star Z = 0$ gives terms of the form $\mathcal{L}_X K_a = K^b \mathcal{L}_X g_{ab}$ ⁽²⁾ which are not necessarily zero

2) taking the rescaled Killing vector field

- $F = d(\psi K)$ ⁽³⁾
- $d \star Z = 0$ gives nonlinear differential equation for ψ with no closed-form solution

→ as these attempts don't work, we will use perturbative expansion around Wald's solution

⁽²⁾ $X^a = \nabla^a \mathcal{F}$

⁽³⁾ ψ is an arbitrary function

Perturbative approach

- expansion of Lagrangian density with respect to coupling constant λ :

$$\mathcal{L}(\mathcal{F}, \mathcal{G}) = -\frac{1}{4}\mathcal{F} + \lambda\ell(\mathcal{F}, \mathcal{G}) + O(\lambda^2)$$

- ansatz: $A_a = K_a + \lambda v_a + O(\lambda^2)$
 v is the lowest order correction to gauge potential
- $F = F_0 + \lambda dv + O(\lambda^2)$
- NLE Maxwell's equations:
 $dF = 0$ - immediately satisfied
 $d \star Z = 0$ - gives master equation for v :

$$d \star dv = \star J_{\text{eff}};$$

$$J_{\text{eff}} = 4(\ell_{\mathcal{F}\mathcal{F}}d\mathcal{F} + \ell_{\mathcal{F}\mathcal{G}}d\mathcal{G})_0 \wedge \star dK - 4(\ell_{\mathcal{G}\mathcal{F}}d\mathcal{F} + \ell_{\mathcal{G}\mathcal{G}}d\mathcal{G})_0 \wedge dK$$

Notation: ℓ_{xy} denotes $\partial_x \partial_y \ell$, subscript 0 means "evaluated at zeroth order of perturbative expansion"

Perturbative approach

We focus on two aforementioned NLE theories:

Born-Infeld

$$\ell^{(\text{BI})} = \mathcal{F}^2 + \mathcal{G}^2$$

$$\lambda^{(\text{BI})} = \frac{1}{32b^2}$$

Euler-Heisenberg

$$\ell^{(\text{EH})} = 4\mathcal{F}^2 + 7\mathcal{G}^2,$$

$$\lambda^{(\text{EH})} = \frac{\alpha^2}{360m_e^4}$$

- both Lagrangians are of the form $\ell = p\mathcal{F}^2 + q\mathcal{G}^2$, so $\ell_{\mathcal{F}\mathcal{G}} = 0$
- master equation reduces to:

$$d \star dv = 4(\ell_{\mathcal{F}\mathcal{F}} d\mathcal{F})_0 \wedge \star dK - 4(\ell_{\mathcal{G}\mathcal{G}} d\mathcal{G})_0 \wedge dK$$

Validity of perturbative approach

- gravitational length scale: L_g
- Orders of magnitude
 - Einstein's tensor: L_g^{-2}
 - magnetic field energy density: $\mathbf{B}^2/(2\mu_0)$
- test field approximation is valid if $L_g^{-2} \gg 4\pi G\mathbf{B}^2/(c^4\mu_0)$
- L_g - Schwarzschild radius $\sim 3(M/M_\odot) \cdot 10^3 m$
- $|\mathbf{B}| \ll (M_\odot/M) \cdot 10^{15} \text{T}$ fulfilled for strongest magnetic fields as long as $M < 10^4 M_\odot$

→ EM field exhibits nonlinear behaviour, but allows test field approximation

Schwarzschild black hole

Schwarzschild spacetime

- static, spherically symmetric vacuum solution

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

for $K^a = \alpha k^a + \beta m^a$, the corresponding invariants are:

- $\mathcal{F}_0 = -\frac{8M^2}{r^4}\alpha^2 + 8\left(1 - \frac{2M}{r}\sin^2\theta\right)\beta^2$
- $\mathcal{G}_0 = -16M\frac{\cos\theta}{r^2}\alpha\beta$

as $\alpha \propto J$ in Wald's solution, $\alpha = 0$

master equation $d \star dv = 4\beta(\ell_{\mathcal{FF}}d\mathcal{F})_0 \wedge \star dm$ gives

$$v = 2\beta^3 M(\ell_{\mathcal{FF}})_0(4(2r - 5M)\cos(2\theta) + (M - 2r)(3 + \cos(4\theta)))d\phi$$

Magnetic scalar potential

"nonlinear H field": $H_a = K^b \star Z_{ba}$

- for symmetry inheriting fields H is a closed form
- we can introduce **scalar magnetic potential** Υ : $H = -d\Upsilon$

expanding with respect to coupling constant λ :

$$\Upsilon = \Psi_0 + \lambda \Psi_1 + O(\lambda^2)$$

from Wald's solution: $\Psi_0 = -B_\infty \left(1 - \frac{2M}{r}\right) r \cos\theta$

Maxwell's equation translates to:

$$\nabla^a \left(\frac{\nabla_a \Psi_1}{k_b k^b} \right) = 16p \nabla^a \left(\frac{(\nabla_c \Psi_0)(\nabla^c \Psi_0)}{(k_b k^b)^2} \nabla_a \Psi_0 \right)$$

solution: $\Psi_1(r, \theta) = 4pB_\infty^3 \left(1 - \frac{2M}{r}\right) (4r - 5M + M \cos(2\theta)) \cos\theta$

→ in agreement with ψ

Asymptotic behaviour

Our solution should:

1) represent asymptotically homogeneous magnetic field

Check:

- F at spatial infinity:

$$\lim_{r \rightarrow \infty} \frac{(dv)_{r\phi}}{(F_0)_{r\phi}} = 0, \quad \lim_{r \rightarrow \infty} \frac{(dv)_{\theta\phi}}{(F_0)_{\theta\phi}} = 0$$

asymptotically,

$$F = F_0 = \frac{1}{2} B_\infty dm = B_\infty (r \sin^2 \theta dr \wedge d\phi + r^2 \cos \theta \sin \theta d\theta \wedge d\phi)$$

- same form as F for homogeneous magnetic field $B_\infty dz$ in Minkowski spacetime

we can choose normalisation of v as $\beta = \frac{1}{2} B_\infty$

2) give $Q = 0$ and $P = 0$

Check:

- Komar integrals for electric and magnetic charge:

$$Q_{\infty} = \frac{1}{4\pi} \oint_{S_{\infty}} *Z$$

- expansion: $*Z = *F_0 + (4(-\ell_{\mathcal{F}}*F + \ell_{\mathcal{G}}F)_0 + *dv)\lambda + O(\lambda^2)$
- $\ell_{\mathcal{F}} = 2\mathcal{F}$, $\ell_{\mathcal{G}} = 2\mathcal{G}$, $\lim_{r \rightarrow \infty} \mathcal{F}_0 = 8\beta^2$
- $(*F_0)_{\theta\varphi} = 0$, $(*dv)_{\theta\varphi} = 0$

→ $Q = 0$ at the $O(\lambda^1)$ order

$$P_{\infty} = \frac{1}{4\pi} \oint_{S_{\infty}} F$$

- $\sin(2\theta)$ and $\sin(4\theta)$ terms in $(dv)_{\theta\varphi}$ vanish after integration

→ $P = 0$ at the $O(\lambda^1)$ order

Correction to \mathcal{F}

Decomposition: $\mathcal{F} = \mathcal{F}_0 + \delta\mathcal{F}$

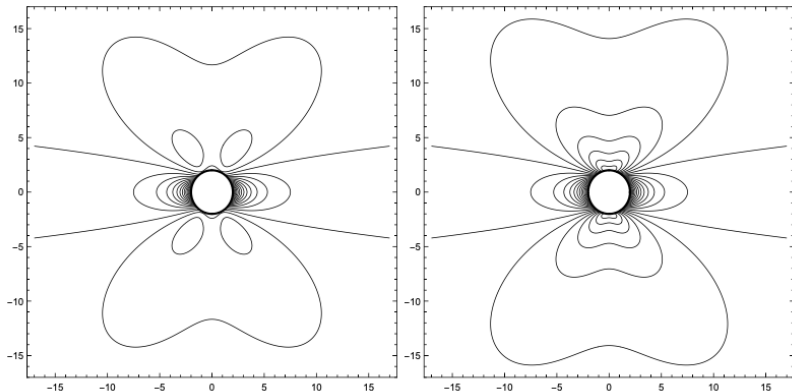
- $\delta\mathcal{F} = -16\lambda B_\infty^4 M(\ell_{\mathcal{FF}})_0 \mathcal{F}_1 + O(\lambda^2),$

$$\mathcal{F}_1 = -\frac{1}{16B_\infty^3 M(\ell_{\mathcal{FF}})_0} (dm)_{ab} (dv)^{ab}$$

Local maxima of \mathcal{F}_1 :

- $r \approx 3.8M, \theta_+ \approx 60.3^\circ$
- $r \approx 3.8M, \theta_- \approx 119.7^\circ$
- implications for trajectories of charged particles?

Contour plots in $r - \theta$ plane



$M = 1$, black hole horizon is represented by the black circle in the middle

Left: Contour plot of \mathcal{F}_1

Right: Contour plot of relative correction $8\beta^2\mathcal{F}_1/\mathcal{F}_0$

Open questions

1) generalisation to non-static, i.e. Kerr black hole

- obstacle: \mathcal{F}_0 and \mathcal{G}_0 have complicated forms and, consequently, current J_{eff}

2) spherically symmetric, highly conducting star immersed in NLE fields

- boundary condition: $n^a \nabla_a \Upsilon = 0$ at star's surface (n^a is a normal to star's boundary)
- we get linear nonhomogeneous partial differential equation for Υ
- obstacle: its solution is an infinite series

References

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