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Optimal Debt Ratio and Dividend Payment Policies for Insurers with Ambiguity

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Abstract: This study considers the optimal debt ratio and dividend payment policies for an insurer concerned about model misspecification. We assume that the insurer can invest all of its asset to the financial market and the ambiguity may exist in the risky asset. Taking into account the ambiguous situation, the insurer aims to maximize the expected utility of a discounted dividend payment until it ruins. Under some assumption, we prove that there exists classical solutions of the optimal debt ratio, dividend payment policies, and value functions that show that the existence of ambiguity can affect the optimal debt ratio and dividend policies significantly.

Keywords: dividend payment; model ambiguity; optimal debt ratio

MSC: 60H30

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1. Introduction

Both dividends and debt are crucial factors in an insurance company. Dividends are a critical component of shareholder returns. Their demonstration shows the company's financial strength and ability to generate profits. Dividends also serve as a signal to investors about the company's future prospects. If the dividend payout ratio is consistent and reliable, it indicates a stable and profitable business model. On the other hand, insurance company debt is also crucial. It is a crucial element in risk management and a key factor in ensuring solvency. The debt-to-asset ratio indicates how well the company manages its balance sheet and risks. A debt ratio that is too high may indicate a leveraged balance sheet and potentially increase the default risk. Conversely, a debt ratio that is too low may indicate underutilized capital and potentially missed opportunities for growth. Therefore, both dividends and debt are important factors to consider when analyzing insurance company performance and financial health.

Due to the nature of their insurance product, insurers sometimes collect substantial sums of cash, cash equivalents, and pursue capital gains in order to cover future claims and prevent bankruptcy. The appropriate debt level and prospective insurance liabilities is of great importance for an insurer. In actuarial science, the appropriate debt level and prospective insurance liabilities of an insurance company should be discussed in detail. Many researchers have investigated the optimal debt policy of an insurer in the last decade. For example, Jin et al. (2015) [1] studied the optimal debt ratio problem considering reinsurance, where they used the subsolution–supersolution method to deal with the existence of solutions of the optimal debt ratio policy. Zhao et al. (2018) [2] considered optimal debt ratio policies for an insurer with a regime-switching model. Qian et al. (2018) [3] investigated the optimal liability ratio under catastrophic risk. In continuous-time setting, Li et al. (2023) [4] researched a state-dependent optimal asset–liability management problem. The optimal debt ratio problem can be seen in other words such as Zhu and Yin (2018) [5], Zhang et al. (2020) [6], Meng and Bi (2020) [7], Abid and Abid (2023) [8], and the references therein.

The seminal work of De Finetti (1957) [9] has led to the classical problem of optimal dividend payment in insurance mathematics. Paulsen and Gjessing (1997) [10] analyzed a risk process with stochastic return on investments and obtained the optimal dividend barrier policy. Cai et al. (2006) [11] investigated the Ornstein–Uhlenbeck-type model with credit and debit interest for the optimal dividend problem. Cheung and Wong (2017) [12] studied the dividend payment in the dual risk model considering delays. Xie and Zhang (2021) [13] researched the finite-time dividend problems in a Levy risk model under periodic observation. Chakraborty et al. (2023) [14] considered a diffusive model for optimally distributing dividends under the situation of Knightian model ambiguity. For more studies on the optimal dividend problem for an insurer, we refer the reader to Avanzi (2009) [15], Yao et al. (2011) [16], Yin and Wen (2013) [17], Bi and Meng (2016) [18], Marciniak and Palmowski (2016) [19], Feng et al. (2021) [20], and so on.

Actuarial research has recently revealed a pattern of diverse development. It is clear that there are growing connections between risk theory and mathematics and some optimum control issues are also becoming more significant and fascinating. As a result, scholars have studied the optimal debt ratio combined with optimal dividend problems in great detail. For example, see Meng et al. (2016) [21], Jin et al. (2022) [22], etc. In the above studies, the scholars did not consider the existence of ambiguity and assumed that the models used are exactly true. In reality, the insurance business uses a wealth of data and a variety of technology to predict actuarial models. It is clear that the insurer is unsure whether a model is the correct model or whether there is a misspecification error. Thus, the aim of this study is to analyze the optimal debt ratio and dividend payment policies for an insurer that is concerned about model misspecification. We assume that the insurer has the ability to invest all of its assets in the financial market and that there may be ambiguity in the risky asset. The insurer's goal is to maximize the expected utility of a discounted dividend payment until it ruins, taking into account the ambiguous situation. Based on some assumptions, we prove that there exists classical solutions of the optimal debt ratio, dividend payment policies, and value functions.

2. Model Formulation

We firstly describe an insurer's surplus process as follows

$$X(t) = K(t) - L(t), \tag{1}$$

where $K(t)$ and $L(t)$ denote asset value and liabilities at time t, respectively. Denote π_t as a debt ratio, i.e.,

$$\pi_t = \frac{L(t)}{X(t)}. \tag{2}$$

Then, we have

$$1 + \pi_t = \frac{K(t)}{X(t)}. \tag{3}$$

Intuitively, if the insurer holds a liability, it will earn premium. Denote α as the premium rate, which means that the insurer can earn α dollars when it has provided a dollar insurance protection. Thus, the increase in the asset value of insurance sales during $[t, t + \mathrm{d}t]$ can be determined by $\alpha L(t)\mathrm{d}t$. Consequently, the insurer aims to know how much of the debt ratio is suitable. For the sake of simplicity, we assume that there is only one risky asset in the financial market. Thus, the price of the risky asset $M(t)$ satisfies that

$$\frac{\mathrm{d}M(t)}{M(t)} = \mu \mathrm{d}t + \sigma \mathrm{d}B_1(t), \tag{4}$$

where $\mu > 0$ and $\sigma > 0$ are real numbers. We assume that the insurer invests all of its asset value $K(t)$ into the financial market. Without considering claims and dividend payment, the surplus process of the insurer can be written as

$$\mathrm{d}X_1(t) = \mathrm{d}K(t) = [\alpha L(t) + K(t)][\mu \mathrm{d}t + \sigma \mathrm{d}B_1(t)]. \tag{5}$$

Then, we assume the accumulated claims up to time t are proportional to the insurer's liabilities $L(t)$, denoted as $S(t)$,

$$S(t) = \int_0^t c(s)L(s)\mathrm{d}t. \tag{6}$$

where $c(t)$ is served as a claim rate which can be described by a diffusion process as

$$\mathrm{d}c(t) = h(c(t))\mathrm{d}t + v\mathrm{d}B_2(t), \tag{7}$$

with $c(0) = c$, where $h(c(t))$: $R \to R$ is an expected claim rate, and $v > 0$ is the volatility of the claim rate. $Cov(B_1(t), B_2(t)) = \rho t$, $-1 < \rho < 1$ represents the correlation between the future claims and the risky asset. Using Gaussian linear regression, we can obtain that

$$\mathrm{d}B_1(t) = \rho \mathrm{d}B_2(t) + \sqrt{1-\rho^2}\mathrm{d}B_3(t), \tag{8}$$

where $B_3(t)$ is a standard Brownian motion, and $B_2(t)$ and $B_3(t)$ are independent.

What is more, we also consider the dividend payment in this paper, and we denote $D(t)$ as accumulated dividend payments up to time t,

$$\mathrm{d}D(t) = z_t X(t)\mathrm{d}t, \ \ with \ \ D(0^-) = 0. \tag{9}$$

where $0 < z_t < M$ is $\mathcal{F}_t$-adapted, and M is a positive constant. Thus, the wealth process of the insurer considering claims and dividend payments can be given in the following

$$\mathrm{d}X(t) = \mathrm{d}X_1(t) - \mathrm{d}S(t) - \mathrm{d}D(t), \ \ X(0) = x. \tag{10}$$

Substituting (5)–(8) into (10), we have

$$\frac{\mathrm{d}X(t)}{X(t)} = (\alpha\pi_t + \pi_t + 1)\mu \mathrm{d}t - c(t)\pi_t \mathrm{d}t - z_t \mathrm{d}t + (\alpha\pi_t + \pi_t + 1)\sigma\rho \mathrm{d}B_2(t) + (\alpha\pi_t + \pi_t + 1)\sigma\sqrt{1-\rho^2}\mathrm{d}B_3(t). \tag{11}$$

The value function is usually set to maximize the expected utility of a discounted dividend payment until it ruins (for example, see Jin et al. (2015) [1]).

$$V_1(x, c) = \sup_{\pi, z} E^P\left[\int_0^\tau U(z_s X(s))\mathrm{d}s \middle| X(0) = x, c(0) = c\right], \tag{12}$$

where τ is the time of ruin and $\tau = \inf\{t \geq 0 : X(t) < 0\}$, where E^P denotes the expectation operator under probability measure P, U is a utility function satisfying $U' > 0$, $U'' < 0$, π, and z are some admissible policies that will be described later. The insurer's understanding of the true probability measure (P) used in computing the equation above is the basic assumption behind this model. The assumption being too strong has been argued by some researchers. Insurers should be permitted to consider optimal policies for other measures of probability. Otherwise, if an insurer ignores the ambiguity of the probability measure and trusts P completely, the insurer may make some mistakes in some decision problems. It is our assumption that the insurer's ambiguity about the financial market model is only due to its limited information about the financial market. The purpose of this paper is to examine the optimal debt ratio and dividend strategy policy with ambiguity in the model against the financial market only. The model ambiguity in our optimal control problem is presented in the following.

We are aware that the probability measure P mentioned above is created using the insurer's limited information. The insurance company computes P by utilizing a vast amount of data and various technologies. This P is referred to as the reference model or reference probability. It is clear that the insurer is unsure whether P is the correct model or whether there was a misspecification error. Naturally, the insurer would take other models into account. We call the aforementioned phenomenon ambiguity. Additionally, we presumptively believe that the model ambiguity is limited to the financial market. The alternative models that the insurer considers should then be identical to the reference model and cannot affect the (7) that corresponds to the claims' arrival rate, so we define the alternative models by a class of probability measures that are equivalent to P:

$$\mathcal{Q} := \{Q|Q \sim P, \ \ Q \text{ will not change } (7)\}, \sim \text{ means equivalent} \tag{13}$$

In a probability space, two measures, P and Q, are equivalent, denoted by $Q \sim P$, if they have same null sets, i.e., $Q(A) = 0$, if and only if $P(A) = 0$)

By Girsanov's theorem (Klebaner (2008) [23]), Q satisfies the following conditions

$$\frac{\mathrm{d}Q}{\mathrm{d}P}(B_3[0,T]) = \Lambda(T), \tag{14}$$

where

$$\Lambda(t) = \exp\left\{\int_0^t m(s)\mathrm{d}B_3(s) - \frac{1}{2}\int_0^t m(s)^2\right\}$$

is a P-martingale with filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, and $m(t)$ is a regular adapted process satisfying Novikov's condition, i.e.,

$$E^P\left[\exp\left(\frac{1}{2}\int_0^T [m(t)]^2\mathrm{d}s\right)\right] < \infty.$$

Then, we have

$$\mathrm{d}B_3(t) = m(t)\mathrm{d}t + \mathrm{d}B_3^Q(t), \tag{15}$$

where $B_3^Q(t)$ is the standard Brownian motion corresponding to the probability measure Q.

We use relative entropy to calculate the difference between each alternative model and the reference model in order to take the alternative model Q into consideration. Relative entropy is a tried-and-true method for calculating the difference between Q and P. Relative entropy has been employed to measure it in numerous research; for examples, see Uppal and Wang (2003) [24], Maenhout (2004) [25], and Yi et al. (2013) [26]. The following is the relative entropy between Q and P.

$$\begin{aligned} H(Q \parallel P) = E^Q\left[\ln\frac{\mathrm{d}Q}{\mathrm{d}P}\right] &= E^Q\left\{\int_0^T m(s)\mathrm{d}B_3(s) - \frac{1}{2}\int_0^T m(s)^2\mathrm{d}s\right\} \\ &= E^Q\left\{\int_0^T m(s)\mathrm{d}B_3^Q(s) + \frac{1}{2}\int_0^T m(s)^2\mathrm{d}s\right\}. \end{aligned}$$

Given that standard Brownian motion is defined by $B_3^Q(t)$ under the probability measure Q, we have

$$H(Q \parallel P) = E^Q\left\{\int_0^T Z(s)\mathrm{d}s\right\} = \int_0^T \frac{1}{2}m(s)^2\mathrm{d}s, \tag{16}$$

where $Z(t) = \frac{1}{2}m(t)^2$. Therefore, the $H(Q \parallel P)$ is measured by $Z(t)$. A penalty is charged if the insurer chooses to use the alternative model Q instead of the reference model P. Naturally, the penalty is increased by the size of the $H(Q \parallel P)$. In a manner similar to Uppal and Wang (2003) [24], the following is how we design a robust control problem:

$$V(x,c)=\sup_{\pi,z\in\mathbb{V}}\inf_{Q\in\mathcal{Q}}E^{Q}_{c,x}\left[\int_0^{\tau}[\xi\phi(V(x,c))Z(s)+U(z_sX(s))\mathrm{d}s]\right],\tag{17}$$

where $E^{Q}_{c,x}[\cdot]=E^{Q}[\cdot|c(0)=c,X(0)=x]$. The standardizing function $\phi(V(t,x))>0$ converts the penalty to the same order of magnitude as $V(x,c)$, where this specific version of $\phi(\cdot)$ is typically chosen for analytical simplicity. The size of the ξ indicates how confident the insurer is in the reference model P, where the larger the ξ is, the more confident the insurer is in P, which we assume as $0<\xi<\infty$ in this paper. The inf term shows the insurer's aversion to ambiguity. In other words, the insurer is conservative and will take into account the worst outcome with ambiguity; it will be further explained that the $\mathbb{V}$ is the set of admissible policies.

3. Main Results

For the purpose of solving (17), the wealth process should be derived under Q. Substituting (15) into (11), we can obtain that

$$\begin{aligned}\frac{\mathrm{d}X(t)}{X(t)} &= (\alpha\pi_t+\pi_t+1)\mu\mathrm{d}t-c(t)\pi_t\mathrm{d}t-z_t\mathrm{d}t+(\alpha\pi_t+\pi_t+1)\sigma m(t)\sqrt{1-\rho^2}\mathrm{d}t\\ &\quad+(\alpha\pi_t+\pi_t+1)\sigma\rho\mathrm{d}B_2(t)+(\alpha\pi_t+\pi_t+1)\sigma\sqrt{1-\rho^2}\mathrm{d}B_3^{Q}(t).\end{aligned}\tag{18}$$

The authors of (18) show that the alternative models only change the drift coefficient, which exactly corresponds to the use of Girsanov's theorem. Let the generator of (18) be

$$\begin{aligned}\mathcal{A}f(x,c) &= \left[(\alpha\pi_t+\pi_t+1)\mu\mathrm{d}t-c\pi_t-z_t+(\alpha\pi_t+\pi_t+1)\sigma m(t)\sqrt{1-\rho^2}\right]x\frac{\partial}{\partial x}f(x,c)\\ &\quad+\frac{1}{2}(\alpha\pi_t+\pi_t+1)^2\sigma^2x^2\frac{\partial^2}{\partial x^2}f(x,c)+h(c)\frac{\partial}{\partial c}f(x,c)+\frac{1}{2}v^2\frac{\partial^2}{\partial c^2}f(x,c)\\ &\quad+(\alpha\pi_t+\pi_t+1)\sigma v\rho x\frac{\partial^2}{\partial c\partial x}f(x,c).\end{aligned}\tag{19}$$

We also provide a definition of the set that includes all admissible policies.

Definition 1. $\mathbb{V}=\{\pi_t,z_t\}$ *is admissible, if*

(i) *The process $z=\{z_t,t\geq 0\}$ is a predictable and satisfy that $0\leq z_t\leq M$;*
(ii) *The process $\pi=\{\pi_t,t\geq 0\}$ is a predictable and satisfy that*

$$E^{Q}\int_0^{T}\pi_s^2ds<\infty,\quad 0<T<\infty,\ \ Q\in\mathcal{Q};$$

(iii) *The stochastic differential Equation (18) determines a unique strong solution.*
Additionally, we state that a pair of policies (π,z) is admissible if $(\pi,z)\in\mathbb{V}$.

It is obvious that $z_t=0$ for $t\geq\tau$. So, we rewrite function (17) as

$$V(x,c)=\sup_{\pi,z\in\mathbb{V}}\inf_{Q\in\mathcal{Q}}E^{Q}_{c,x}\left[\int_0^{\infty}[\xi\phi(V(x,c))Z(s)+U(z_sX(s))\mathrm{d}s]\right],\tag{20}$$

The Hamilton–Jacobi–Bellman(HJB) (See Fleming and Soner (2006) [27]) equation, which is satisfied by the value function (20), can be given as follows.

$$\max_{\pi,z}\inf_{m}\left\{\mathcal{A}V(x,c)-rV(x,c)+U(z_tx)+\frac{1}{2}\xi\phi(V(x,c))m^2\right\}=0\tag{21}$$

Let $G_t := \alpha\pi_t + \pi_t + 1$, then $\pi_t = \frac{G_t - 1}{\alpha+1}$. Additionally, as previously noted, we select a suitable form of $\phi(\cdot)$ (the form has been employed in Uppal and Wang (2003) [24] and other instances),

$$\phi(V(x,c)) = V(x,c).$$

Hence, the Equation (21) can be simplified as

$$\begin{aligned}\max_{G,z}\inf_{m} \quad & \Bigg\{\Bigg[G_t\mu dt - \frac{c(G_t-1)}{\alpha+1} - z_t + G_t\sigma m(t)\sqrt{1-\rho^2}\Bigg]x\frac{\partial}{\partial x}V(x,c) \\ & + \frac{1}{2}G_t^2\sigma^2x^2\frac{\partial^2}{\partial x^2}V(x,c) + h(c)\frac{\partial}{\partial c}V(x,c) + \frac{1}{2}v^2\frac{\partial^2}{\partial c^2}V(x,c) \\ & + G_t\sigma v\rho x\frac{\partial^2}{\partial c\partial x}V(x,c) - rV(x,c) + U(z_tx) + \frac{1}{2}\xi V(x,c)m^2\Bigg\} = 0 \end{aligned} \tag{22}$$

Since $\frac{1}{2}\xi V(x,c) > 0$, the function m^* minimizes (22) according to the first-order condition, which takes the following form.

$$m^* = \frac{-G_t\sigma\sqrt{1-\rho^2}x\frac{\partial}{\partial x}V(x,c)}{\xi V(x,c)}. \tag{23}$$

Substituting (23) into (22) yields

$$\begin{aligned}&\max_{G}\Bigg\{\frac{1}{2}G_t^2\sigma^2x^2\frac{\partial^2}{\partial x^2}V(x,c) - \frac{G_t^2\sigma^2(1-\rho^2)x^2[\frac{\partial}{\partial x}V(x,c)]^2}{2\xi V(x,c)} + [G_t\mu - \frac{cG_t}{\alpha+1}]x\frac{\partial}{\partial x}V(x,c) \\ &+ G_t\sigma v\rho x\frac{\partial^2}{\partial c\partial x}V(x,c)\Bigg\} + \max_{z}\Bigg\{-z_tx\frac{\partial}{\partial x}V(x,c) + U(z_tx)\Bigg\} \\ &+ \frac{1}{2}v^2\frac{\partial^2}{\partial c^2}V(x,c) + h(c)\frac{\partial}{\partial c}V(x,c) - rV(x,c) + \frac{cx}{\alpha+1}\frac{\partial}{\partial x}V(x,c) = 0 \end{aligned} \tag{24}$$

Assume that the utility function has the following form

$$U(x) = \frac{x^\gamma}{\gamma}, \tag{25}$$

where $0 < \gamma < 1$. We speculate that the value function has the following form given the utility function.

$$V(x,c) = Y(c)\frac{x^\gamma}{\gamma}, \tag{26}$$

where $Y(c)$ is a function of c. To determine $Y(c)$, we derive the following functions

$$\begin{cases} \frac{\partial}{\partial x}V(x,c) = Y(c)x^{\gamma-1}, \\ \frac{\partial^2}{\partial x^2}V(x,c) = Y(c)(\gamma-1)x^{\gamma-2}, \\ \frac{\partial}{\partial c}V(x,c) = Y'(c)\frac{x^\gamma}{\gamma}, \\ \frac{\partial^2}{\partial c^2}V(x,c) = Y''(c)\frac{x^\gamma}{\gamma}, \\ \frac{\partial^2}{\partial x\partial c}V(x,c) = Y'(c)x^{\gamma-1}. \end{cases} \tag{27}$$

Substituting (27) into (24) and simplifying it, we can obtain that

$$\max_{G}\left\{\frac{1}{2}G_t^2[\sigma^2(\gamma-1)Y(c)-\frac{\gamma}{\xi}\sigma^2(1-\rho^2)Y(c)]+G_t[(\mu-\frac{c}{\alpha+1})Y(c)+\sigma\upsilon\rho Y'(c)]\right\}$$
$$+\max_{z}\left\{-z_tY(c)+\frac{z_t^{\gamma}}{\gamma}\right\}+\frac{1}{2}\upsilon^2Y''(c)\frac{1}{\gamma}+h(c)Y'(c)\frac{1}{\gamma}-rY(c)\frac{1}{\gamma}+\frac{c}{\alpha+1}Y(c)=0. \tag{28}$$

According to the first-order conditions, we can obtain that

$$\begin{cases} G_t* = \dfrac{(\mu-\frac{c}{\alpha+1})Y(c)+\sigma\upsilon\rho Y'(c)}{\sigma^2 Y(c)[(1-\gamma)+\frac{(1-\rho^2)\gamma}{\xi}]}, \\ z_t^* = Y(c)^{\frac{1}{\gamma-1}}. \end{cases} \tag{29}$$

Substituting (29) into (28), we have

$$\frac{1}{2}\frac{[(\mu-\frac{c}{\alpha+1})Y(c)+\sigma\upsilon\rho Y'(c)]^2}{\sigma^2Y(c)[(1-\gamma)+\frac{(1-\rho^2)\gamma}{\xi}]}+(\frac{1}{\gamma}-1)Y(c)^{\frac{\gamma}{\gamma-1}}$$
$$+\frac{1}{2}\upsilon^2Y''(c)\frac{1}{\gamma}+h(c)Y'(c)\frac{1}{\gamma}-rY(c)\frac{1}{\gamma}+\frac{c}{\alpha+1}Y(c)=0. \tag{30}$$

Multiplying both sides by γ in the above equation and simplifying it, (30) can be represented as

$$\frac{1}{2}\nu^2Y''(c)+[h(c)+\frac{\nu\rho\gamma(\mu-\frac{c}{\alpha+1})}{\sigma[(1-\gamma)+\frac{\gamma(1-\rho^2)}{\xi}]}]Y'(c)+[\frac{1}{2}\frac{\gamma(\mu-\frac{c}{\alpha+1})^2}{\sigma^2[(1-\gamma)+\frac{(1-\rho^2)\gamma}{\xi}]}+\frac{c\gamma}{\alpha+1}-r]Y(c)$$
$$+\frac{1}{2}\frac{\nu^2\rho^2\gamma}{(1-\gamma)+\frac{(1-\rho^2)\gamma}{\xi}}\frac{Y'(c)^2}{Y(c)}+(1-\gamma)Y(c)^{\frac{\gamma}{\gamma-1}}=0. \tag{31}$$

For the sake of simplicity, denote $\Lambda(c)=\ln Y(c)$. Additionally, let

$$\begin{cases} H(c) = h(c)+\dfrac{\nu\rho\gamma(\mu-\frac{c}{\alpha+1})}{\sigma[(1-\gamma)+\frac{\gamma(1-\rho^2)}{\xi}]}, \\ K(c) = \dfrac{1}{2}\dfrac{\gamma(\mu-\frac{c}{\alpha+1})^2}{\sigma^2[(1-\gamma)+\frac{(1-\rho^2)\gamma}{\xi}]}+\dfrac{c\gamma}{\alpha+1}, \\ L = \dfrac{1}{2}\dfrac{\nu^2\rho^2\gamma}{(1-\gamma)+\frac{(1-\rho^2)\gamma}{\xi}}. \end{cases} \tag{32}$$

Then (31) can be represented as

$$\frac{1}{2}\nu^2\Lambda''(c)+H(c)\Lambda'(c)+[L+\frac{\nu^2}{2}]\Lambda'(c)^2+N(c)=0, \tag{33}$$

where $N(c)=K(c)-r+(1-\gamma)e^{\frac{\Lambda(c)}{\gamma-1}}$. Next, we will verify the existence of the classical solution of $\Lambda(c)$. Naturally, from $Y(c)=e^{\Lambda(c)}$, we can also obtain $Y(c)$. In order to obtain the classical solution of (33), we apply the method that Jin et al. (2015) [1] used named subsolution and supersolution. The definition of subsolution and supersolution can be presented in the following.

Definition 2. *A solution $\Lambda_1(c)$ is said to be a subsolution of (33) if $\forall c \in R$, $\Lambda_1(c) \in C^2(R)$ and $\Lambda_1(c)$ satisfies*

$$\frac{1}{2}\nu^2\Lambda_1''(c) + H(c)\Lambda_1'(c) + [L(c) + \frac{\nu^2}{2}]\Lambda_1'(c)^2 + N(c) \geq 0 \tag{34}$$

A solution $\Lambda_2(c)$ is said to be a supersolution of (33) if $\forall\, c \in R$, $\Lambda_2(c) \in C^2(R)$ and $\Lambda_2(c)$ satisfies

$$\frac{1}{2}\nu^2\Lambda_2''(c) + H(c)\Lambda_2'(c) + [L(c) + \frac{\nu^2}{2}]\Lambda_2'(c)^2 + N(c) \leq 0. \tag{35}$$

Furthermore, if $\forall c \in R$ $\Lambda_1(c) \leq \Lambda_2(c)$, we say $\Lambda_1(c)$ and $\Lambda_2(c)$ are an ordered pair of subsolution and supersolution, respectively.

In order to obtain the existence of the classical solution of (33), we give the following lemmas.

Lemma 1. *Suppose that*

$$r > \gamma[\mu - \frac{\sigma^2(1-\gamma) + \frac{(1-\rho^2)\gamma}{\xi}}{2}], \tag{36}$$

then

$$\underline{\Lambda} = (\gamma - 1)\ln\left\{\frac{2}{\gamma - 1}[r - \gamma(\mu - \frac{\sigma^2(1-\gamma) + \frac{(1-\rho^2)\gamma}{\xi}}{2})]\right\} \tag{37}$$

is a subsolution of (33).

Proof. Since

$$\begin{aligned}
K(c) =&\frac{1}{2}\frac{\gamma(\mu - \frac{c}{\alpha+1})^2}{\sigma^2[(1-\gamma) + \frac{(1-\rho^2)\gamma}{\xi}]} + \frac{c\gamma}{\alpha+1} \\
=&\frac{\gamma\left(\mu - \frac{c}{\alpha+1} - \sigma^2[(1-\gamma) + \frac{(1-\rho^2)\gamma}{\xi}]\right)^2}{2\sigma^2[(1-\gamma) + \frac{(1-\rho^2)\gamma}{\xi}]} + \gamma\mu - \frac{\gamma\sigma^2[(1-\gamma) + \frac{(1-\rho^2)\gamma}{\xi}]}{2} \\
\geq&\gamma\mu - \frac{\gamma\sigma^2[(1-\gamma) + \frac{(1-\rho^2)\gamma}{\xi}]}{2},
\end{aligned} \tag{38}$$

hence

$$\begin{aligned}
N(c) =&K(c) - r + (1-\gamma)e^{\frac{\Lambda(c)}{\gamma-1}} \geq \gamma\mu - \frac{\gamma\sigma^2[(1-\gamma) + \frac{(1-\rho^2)\gamma}{\xi}]}{2} - r + 2[r - \gamma(\mu - \frac{\sigma^2(1-\gamma) + \frac{(1-\rho^2)\gamma}{\xi}}{2})] \\
=&r - [\gamma\mu - \frac{\gamma\sigma^2[(1-\gamma) + \frac{(1-\rho^2)\gamma}{\xi}]}{2}] > 0.
\end{aligned} \tag{39}$$

Combining with $\frac{1}{2}\nu^2\underline{\Lambda}'' + H(c)\underline{\Lambda}' + [L + \frac{\nu^2}{2}]\underline{\Lambda}'^2 = 0$, we complete the proof. □

Let

$$\begin{aligned}
l_1 &= 2\nu^2(\frac{\gamma\rho^2}{1-\gamma+\frac{\gamma(1-\rho^2)}{\xi}} + 1), \\
l_2 &= -2\frac{\nu\rho\gamma\frac{1}{\alpha+1}}{\sigma[(1-\gamma) + \frac{\gamma(1-\rho^2)}{\xi}]} + 2\kappa, \\
l_3 &= \frac{\gamma}{2\sigma^2(1-\gamma+\frac{(1-\rho^2)\gamma}{\xi})(\alpha+1)^2},
\end{aligned}$$

where $\kappa > \frac{\nu\rho\gamma}{\sigma[(1-\gamma)+\frac{\gamma(1-\rho^2)}{\xi}](\alpha+1)} - \frac{\nu\sqrt{\gamma[\gamma\rho^2+1-\gamma+\frac{\gamma(1-\rho^2)}{\xi}]}}{\sigma[1-\gamma+\frac{(1-\rho^2)\gamma}{\xi}](\alpha+1)}$. Then the equation $l_1k^2 + l_2k + l_3 = 0$ has two positive real roots denoted by k_1 and k_2.

Obviously, $l_1, l_3 > 0$, $l_2 < 0$ and $l_2^2 - 4l_1l_3 > 0$. So the equation has two positive roots.

Lemma 2. *Let $k_0 = \frac{k_1+k_2}{2}$, then $k_0 > 0$. Additionally, assume that $h'(c) < \kappa$ and*

$$r > K_1(k_0), \tag{40}$$

where $K_1(k_0)$ will be given later. Then

$$\tilde{\Lambda}(c) = k_0c^2 + K_0 \tag{41}$$

is a supersolution of (33), where K_0 is a constant which is large enough such that $\tilde{\Lambda}(c) > \Lambda_1$ and satisfies that $K_0 > (\gamma-1)\ln\left(\frac{r-K_1(k_0)}{1-\gamma}\right)$.

Proof. From (33) and (41), we have

$$\begin{aligned}
&\frac{1}{2}\nu^2\tilde{\Lambda}''(c) + H(c)\tilde{\Lambda}'(c) + [L + \frac{\nu^2}{2}]\tilde{\Lambda}'(c)^2 \\
= \quad & k_0\nu^2 + 2H(c)k_0c + [L + \frac{\nu^2}{2}](2k_0c)^2 \\
= \quad & k_0\nu^2 + [2h(c) + 2\frac{\nu\rho\gamma(\mu - \frac{c}{\alpha+1})}{\sigma[(1-\gamma) + \frac{\gamma(1-\rho^2)}{\xi}]}]k_0c + [\frac{\nu^2\rho^2\gamma}{(1-\gamma)+\frac{(1-\rho^2)\gamma}{\xi}} + \nu^2]2k_0^2c^2 \\
= \quad & 2k_0^2c^2\nu^2[\frac{\rho^2\gamma}{(1-\gamma)+\frac{(1-\rho^2)\gamma}{\xi}} + 1] + k_0\nu^2 + 2k_0c[\frac{\nu\rho\gamma(\mu - \frac{c}{\alpha+1})}{\sigma[(1-\gamma) + \frac{\gamma(1-\rho^2)}{\xi}]}] + 2h(c)k_0c
\end{aligned} \tag{42}$$

We know that $\exists\ \hat{c}$ s.t.

$$h(c) = h(0) + ch'(\hat{c}) < h(0) + c\kappa. \tag{43}$$

Hence,

$$\begin{aligned}
&\frac{1}{2}\nu^2\tilde{\Lambda}''(c) + H(c)\tilde{\Lambda}'(c) + [L + \frac{\nu^2}{2}]\tilde{\Lambda}'(c)^2 \\
< \quad & 2k_0^2c^2\nu^2[\frac{\rho^2\gamma}{(1-\gamma)+\frac{(1-\rho^2)\gamma}{\xi}} + 1] + k_0\nu^2 + 2k_0c[\frac{\nu\rho\gamma(\mu - \frac{c}{\alpha+1})}{\sigma[(1-\gamma) + \frac{\gamma(1-\rho^2)}{\xi}]}] + 2(h(0) + c\kappa)k_0c. \\
= \quad & c^2(k_0^2l_1 + k_0l_2) + 2k_0c[\frac{\nu\rho\gamma\mu}{\sigma[(1-\gamma) + \frac{\gamma(1-\rho^2)}{\xi}]} + h(0)] + k_0\nu^2.
\end{aligned} \tag{44}$$

So

$$
\begin{aligned}
& \frac{1}{2}\nu^2\tilde{\Lambda}''(c)+H(c)\tilde{\Lambda}'(c)+[L+\frac{\nu^2}{2}]\tilde{\Lambda}'(c)^2+N(c) \\
< \quad & c^2(k_0^2 l_1+k_0 l_2)+2k_0 c[\frac{\nu\rho\gamma\mu}{\sigma[(1-\gamma)+\frac{\gamma(1-\rho^2)}{\xi}]}+h(0)]+k_0\nu^2 \\
& +\frac{1}{2}\frac{\gamma(\mu-\frac{c}{\alpha+1})^2}{\sigma^2[(1-\gamma)+\frac{(1-\rho^2)\gamma}{\xi}]}+\frac{c\gamma}{\alpha+1}-r+(1-\gamma)e^{\frac{\tilde{\Lambda}(c)}{\gamma-1}} \\
= \quad & c^2(k_0^2 l_1+k_0 l_2)+2k_0 c[\frac{\nu\rho\gamma\mu}{\sigma[(1-\gamma)+\frac{\gamma(1-\rho^2)}{\xi}]}+h(0)]+k_0\nu^2 \\
& +\frac{1}{2}\frac{\gamma\Big\{\frac{c^2}{(\alpha+1)^2}+\{\mu-\sigma^2[1-\gamma+\frac{(1-\rho^2)\gamma}{\xi}]\}^2-\frac{2c}{\alpha+1}\{\mu-\sigma^2[1-\gamma+\frac{(1-\rho^2)\gamma}{\xi}]\}\Big\}}{\sigma^2[(1-\gamma)+\frac{(1-\rho^2)\gamma}{\xi}]} \\
& +\gamma(\mu-\frac{c}{\alpha+1})-\frac{\gamma\sigma^2[1-\gamma+\frac{(1-\rho^2)\gamma}{\xi}]}{2}+\frac{c\gamma}{\alpha+1}-r+(1-\gamma)e^{\frac{\tilde{\Lambda}(c)}{\gamma-1}} \\
= \quad & c^2\varphi_1^2+c\varphi_2+\varphi_3-r+(1-\gamma)e^{\frac{\tilde{\Lambda}(c)}{\gamma-1}},
\end{aligned} \tag{45}
$$

where

$$
\begin{aligned}
\varphi_1 &= k_0^2 l_1+k_0 l_2+l_3, \\
\varphi_2 &= 2k_0[\frac{\nu\rho\gamma\mu}{\sigma[(1-\gamma)+\frac{\gamma(1-\rho^2)}{\xi}]}+h(0)]-\frac{\gamma[\mu-\sigma^2[(1-\gamma)+\frac{(1-\rho^2)\gamma}{\xi}]]}{\sigma^2[(1-\gamma)+\frac{(1-\rho^2)\gamma}{\xi}](\alpha+1)}, \\
\varphi_3 &= k_0\nu^2+\frac{1}{2}\frac{\gamma\{\mu-\sigma^2[1-\gamma+\frac{(1-\rho^2)\gamma}{\xi}]\}^2}{\sigma^2[(1-\gamma)+\frac{(1-\rho^2)\gamma}{\xi}]}+\gamma\mu-\frac{\gamma\sigma^2[1-\gamma+\frac{(1-\rho^2)\gamma}{\xi}]}{2}.
\end{aligned}
$$

Let $K_1(k_0)=\varphi_3-\frac{\varphi_2^2}{4\varphi_1}$, we obtain that

$$
\begin{aligned}
& \frac{1}{2}\nu^2\tilde{\Lambda}''(c)+H(c)\tilde{\Lambda}'(c)+[L+\frac{\nu^2}{2}]\tilde{\Lambda}'(c)^2+N(c) \\
< \quad & K(k_0)-r+(1-\gamma)e^{\frac{k_0c^2+K_0}{\gamma-1}} \\
< \quad & (r-K(k_0))[e^{\frac{k_0c^2}{\gamma-1}}-1]<0
\end{aligned} \tag{46}
$$

□

By Lemmas 1 and 2 , we have Theorem 1.

Theorem 1. *There exists a classical solution of* (33) *denoted by* $\hat{\Lambda}(c)$ *such that*

$$
\underline{\Lambda}\le\hat{\Lambda}(c)\le\tilde{\Lambda}(c). \tag{47}
$$

Proof. An ordered pair of subsolution and supersolution of (42) are obtained by Lemmas 1 and 2. Then the existence of a classical solution can be proved by Theorem 3.4 in Jin et al. (2015) [1]. □

Then, the value function, optimal debt ratio strategy and optimal dividend strategy are given as follows.

Theorem 2. *Suppose that a function $\hat{\Lambda}(c)$ solves (33), then there exists $\hat{Y}(c)$ that solves (31). Additionally, assume that (36) and (40) hold. Then,*

(i)

$$\hat{V}(x,c) = \hat{Y}(c)\frac{x^{\gamma}}{\gamma}, \tag{48}$$

is the value function of (20);

(ii) The optimal debt ratio and optimal dividend policies are given by

$$\begin{cases} \pi_t^* = \dfrac{G_t^* - 1}{\alpha + 1}, \\ z_t^* = \hat{Y}(c)^{\frac{1}{\gamma-1}}, \end{cases} \tag{49}$$

where $G_t^* = \frac{(\mu - \frac{c}{\alpha+1})\hat{Y}(c) + \sigma\nu\rho\hat{Y}'(c)}{\sigma^2\hat{Y}(c)[(1-\gamma)+\frac{(1-\rho^2)\gamma}{\xi}]}$.

Remark 1. *We can see that in Theorem 2, the optimal policies and the value function can be affected by the ambiguity parameter ξ, which means that the existence of the ambiguity can affect the optimal debt ratio and dividend policies and that insurers cannot ignore the existence of ambiguity when making their decisions.*

4. Conclusions

In the modern field of actuarial science, optimal debt ratio decisions and dividend problems are extremely important. Most of the existing works only deal with this interesting topic under the assumption of an accurate model. We investigate the optimal debt ratio and dividend payment policies for an insurer concerned about model misspecification and prove that there exists classical solutions of the optimal debt ratio, dividend payment policies, and value functions.

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