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*Article*

# About Oscillations in Nonlinear Systems with Elastic Bonds

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**Abstract:** This article deals with oscillations that occur with a certain combination of parameters in a nonlinear system with elastic bonds, formed by three cylinders pulled together by an elastic thread. The aim of this work was to develop a methodology for studying nonlinear oscillatory systems based on the analysis of the potential energy of a system and the balance of the forces acting on it. The novelty of the work lies in the proposed methods for determining the conditions for the occurrence of vibrations and in those for calculating the threshold value of the elasticity coefficient of an elastic thread, at which an oscillatory process is guaranteed to occur. The differential equations of oscillations were compiled and numerically solved both with and without allowance for friction forces. The critical value of the elasticity coefficient of the thread at which periodic oscillations occurred was determined. A study of the motion of the system was carried out.

**Keywords:** non-linear oscillatory systems; elastic bonds

**MSC:** 37M05

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## 1. Introduction

Over the past decades, a large number of various methods for studying nonlinear dynamic systems, in particular, oscillatory systems, have been developed [1–14]. One of the most relevant and widely used is the method of numerical modeling [15–19]. The advantages of this method are quite obvious: it not only allows one to study systems for a wide range of different parameters, but also often reveals (sometimes accidentally) unexpected effects that can be hidden in analytical analyses. The prevalence of numerical modeling is also associated with the wide computing capabilities of modern mathematical computer packages [20–22]. In addition, numerical modeling makes it possible to evaluate the effectiveness of various methods for studying nonlinear systems, for example, whether to use the classical method of compiling and analytically solving a differential equation or to use difference schemes for describing the process [23].

Methods for studying nonlinear systems are very diverse and use, for example, both the classical Lagrange technique [24,25] and differential analysis [26], a novel machine learning technique [27] or a path-following method [28].

In this paper, we consider the case of an "unexpected" oscillatory process that occurs with a certain combination of parameters in a nonlinear dynamic system of three cylinders pulled together by an elastic thread. The main purpose of this work was to develop a methodology for the study of nonlinear oscillatory systems, using the analysis of the potential energy of the system and the balance of the forces acting on it. The novelty of this work lies in the proposed methods for determining the conditions for the occurrence of vibrations and in those for calculating the threshold value of the elasticity coefficient of an elastic thread, at which an oscillatory process is guaranteed to occur.

With the help of mathematical modeling using the Mathcad computer package, a numerical solution of differential equations describing the oscillatory process was obtained both with and without regard to friction forces.

Despite the apparent triviality of the system under consideration, the formulation of differential equations describing its motion and their numerical solutions are not a trivial task but require thoughtful analysis, taking into account various factors and the use of a number of tools of the Mathcad package.

The article has the following structure:

- design a mathematical model of the system in the absence of friction;
- analyze the conditions for the occurrence of oscillations in a frictionless system and determine the critical value of the coefficient of elasticity;
- design a mathematical model of the system in the presence of friction;
- study the behavior of the system under various initial conditions and values of the friction coefficient.

## 2. Materials and Methods

This work used methods of mathematical modeling, methods for the numerical solution of differential equations, methods for solving algebraic equations in Mathcad [29], optimization methods, experimental methods.

## 3. Results and Discussion

### *3.1. Mathematical Model of a Pendulum without Friction Forces*

Consider three cylinders of mass $m$ and radius $r$, which are placed side by side on a flat surface and then pulled together by an elastic thread (rubber band) with an elasticity coefficient $k$ (Figure 1a). If the rubber band has sufficient rigidity, then the middle of the cylinders will move up, and the system will take the position shown in Figure 1b.

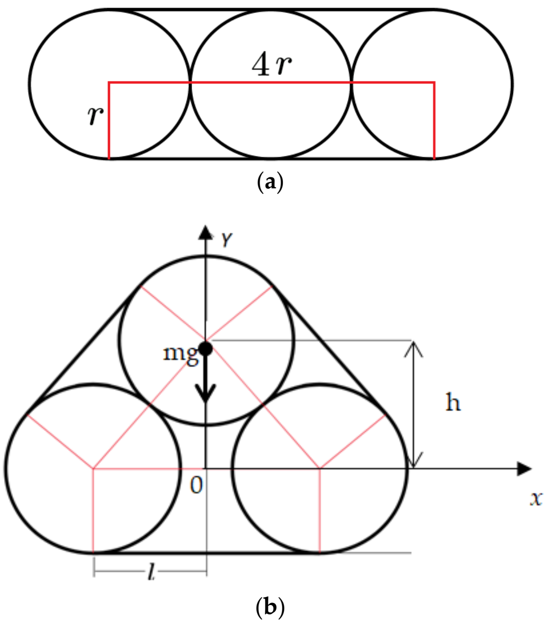


**Figure 1.** Calculation scheme of the pendulum (excluding friction forces). (**a**,**b**) has been explained in the paragraph above.

The middle (upper) cylinder can be in a stable position, shown in Figure 1b and characterized by the fact that the sum of the potential energies of the upper cylinder and the stretched rubber band will be minimal (the principle of minimum total potential energy [30,31]).

Indeed, assuming that the elastic thread is stretched according to the linear Hooke's law [32], from simple geometric considerations (Figure 1b), we can calculate the length of the elastic thread $L$ (note that this case can also be generalized for the nonlinear dependence of the elasticity coefficient [33]):

$$\begin{aligned} l &= \sqrt{(2r)^2 - h^2}, \\ L &= 2(l + \pi r + 2r). \end{aligned} \tag{1}$$

Let us compare $L$ with the length $L_0$ of an unstretched thread holding the cylinders in a closed position, shown in Figure 2:

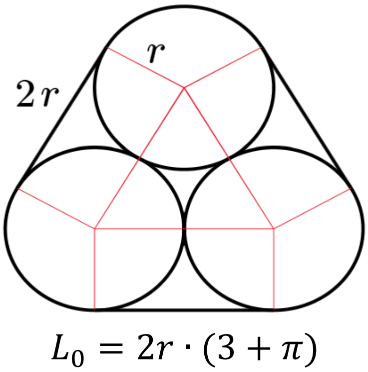


**Figure 2.** Cylinders connected by an inelastic thread.

Thus, the stretch of the thread is:

$$\Delta L = L - 2r(3 + \pi) = 2\left(\sqrt{4\,r^2 - h^2} - r\right). \tag{2}$$

The potential energy $E_p$ will be determined as follows:

$$E_p = mgh + \frac{k\Delta L^2}{2}. \tag{3}$$

By specifying the values of the mass and radius of the cylinder, $m$ = 1 kg, $r$ = 1 m, and choosing the value of the elasticity coefficient, $k$ = 6.5 N/m, we could obtain the dependence of the potential energy $E_p$ on the value $h$, which varied in the range from 0 to $r\sqrt{3}$ (Figure 3).

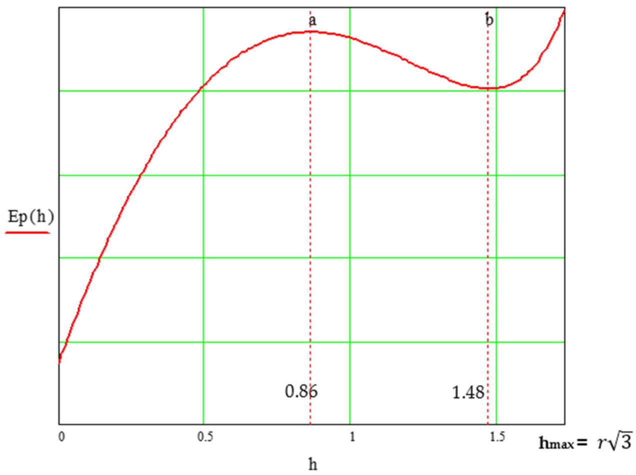


**Figure 3.** Potential energy graph for $k$ = 6.5 N/m.

On the graph presented in Figure 3, we observe a local minimum of potential energy at the point $h$ = $b$ = 1.48 m, which corresponds to the stable position of the middle cylinder.

The presence of such a local minimum between neighboring maxima at the points $h$ = $a$ = 0.86 m and $h = r\sqrt{3}$ led us to the question: can oscillations occur in the vicinity of a stable position when the system is perturbed? Physically, this can be interpreted as follows: if we move ("pull") the middle cylinder up and release it, how will the system behave?

To answer this question, we composed a differential equation that describes the movement of the middle cylinder, where the desired function is $y(t)$—the coordinate of the center of the middle cylinder. To do this, we used the law of conservation of the total mechanical energy of the entire system of three cylinders [30] (we neglected friction, first).

The potential energy of the system can be written as:

$$E_p = mgy(t) + 2k\left(5r^2 - y(t)^2 \; - 2r\sqrt{4\,r^2 - y(t)^2}\right). \tag{4}$$

Let us create an expression for the kinetic energy of all three cylinders.

The kinetic energy of the middle cylinder is defined as

$$E_{k1} = \frac{m(y'(t))^2}{2}, \tag{5}$$

and the kinetic energy of each of the other two cylinders is defined as

$$E_{k23} = \frac{m}{2}\left(\frac{d\left(2r - \sqrt{4\,r^2 - y(t)^2}\right)}{dt}\right)^2. \tag{6}$$

Then, the total kinetic energy of the system after simplification is:

$$E_k = \frac{m(y'(t))^2}{2} \cdot \frac{4\,r^2 + y(t)^2}{4\,r^2 - y(t)^2}. \tag{7}$$

By virtue of the law of conservation of the total mechanical energy of a system without friction, we obtain

$$\frac{d\left(E_{p+}E_k\right)}{dt} = 0. \tag{8}$$

Then, substituting expressions (4) and (7) into (8), we finally obtain a second-order nonlinear differential equation describing the motion of the middle cylinder:

$$2m\frac{A}{B}y''(t) = 2m\frac{2A-B}{B^2}\left(y'(t)\right)^2 y(t) - 4ky(t)\left(1 - \frac{r}{\sqrt{4\,r^2 - y(t)^2}}\right) + mg, \tag{9}$$

where

$$A = 4r^2 + y(t)^2, \;\; B = 2y(t)^2 - 8r^2. \tag{10}$$

The result of the numerical solution of the differential Equation (9) in the Mathcad package with the above values $m$ =1 kg, $r$ = 1 m, $k$ = 6.5 N/m and initial conditions

$$y(0) = r, \;\; y'(0) = 0, \tag{11}$$

is shown in Figure 4.

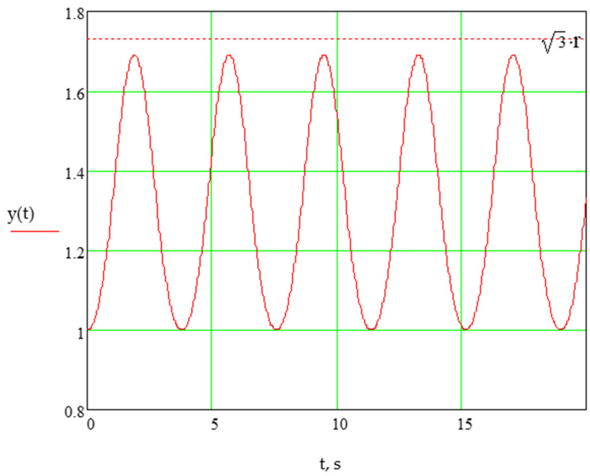


**Figure 4.** Solution of differential Equation (9).

Based on the graph in Figure 4, we really observed periodic oscillations that the middle cylinder made around the equilibrium position when the system was perturbed (without taking into account friction forces), so that, in fact, it resembled as a kind of pendulum.

*3.2. Analysis of the Conditions for the Occurrence of Periodic Oscillations*

The solution presented in Figure 4 was obtained with a specially selected value of the coefficient of elasticity $k$. From physical considerations, it is clear that if the value of the coefficient of elasticity is small, then the middle cylinder, being forcefully raised above the surface, will again fall on it, that is, it will return to the position shown in Figure 1a. That is, periodic oscillations can occur only at certain elasticity coefficients that are above a certain threshold level.

This could be clearly demonstrated by plotting the dependence of the potential energy of the middle cylinder $E_p$ vs. $h$, with the elasticity coefficient $k$ with values from 0.5 to 6.5 N/m (Figure 5).

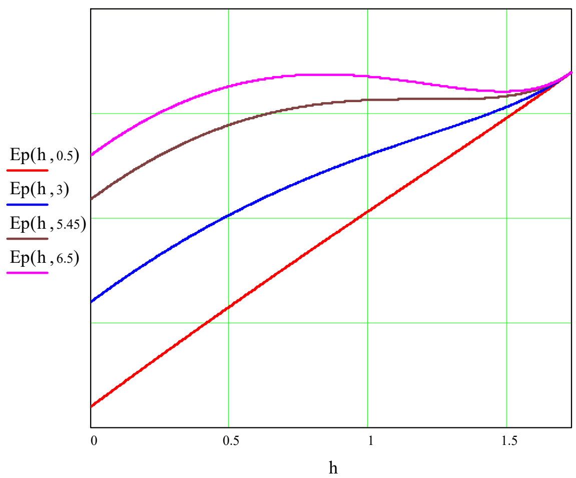


**Figure 5.** Graph of the potential energy of the average cylinder for different $k$.

Clearly, on the graphs of Figure 5 for $k$ with values of 0.5; 3; 5.45 N/m, we do not observe a local minimum of potential energy, which means that there was no possibility of periodic oscillations for those values of $k$.

Thus, we needed to determine the lower limit of the elasticity coefficient $k$, above which periodic oscillations would occur. For empirical reasons, we looked for this boundary in the form

$$k = q \cdot mg/r, \tag{12}$$

where the dimensionless quantity $q$ is the desired threshold value.

Let us use consider the following. We needed the potential energy to reach its local minimum, that is, that its derivative be equal to 0. Let us find the derivative of the potential energy $E_p$ with respect to $h$, after substituting (1) and (2) into (3):

$$E'_p = mg + \frac{2kh\left(2r - 2\sqrt{4\,r^2 - h^2}\right)}{\sqrt{4\,r^2 - h^2}}. \tag{13}$$

Then, after substituting (12) into (13) and equating to 0, we finally obtain an irrational equation with parameter $q$ and variable $h$:

$$\left(1 - \frac{4q}{r}h\right)\sqrt{4\,r^2 - h^2} + 4qh = 0. \tag{14}$$

To analyze the equation, we plotted the left side of Equation (14) $Eq\ (q, h)$ vs. the values $q$ = 0.2; 0.8 with a fixed $r$ = 1 (Figure 6).

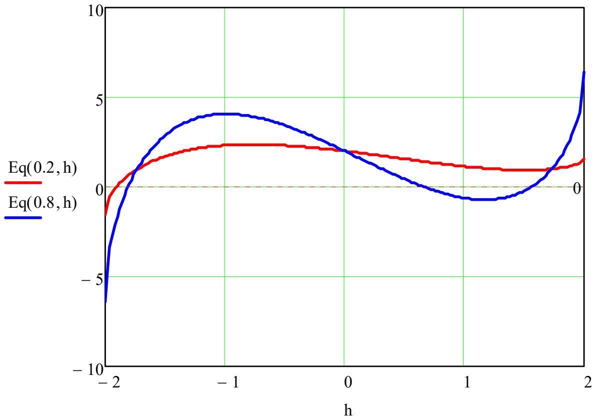


**Figure 6.** Graph of the left side of Equation (14).

Figure 6 shows that the required solution was achieved if (14) had three real roots, two of which were positive (as in the case of $Eq\ (0.8, h)$). These roots corresponded to the points of local maximum $a$ and minimum $b$ (Figure 3). That is, we needed to determine the threshold value $q$ for values above which Equation (14) was solvable in real numbers only.

To do this, we numerically solved Equation (14) in the Mathcad package for a fixed value $r$, changing the parameter $q$ from 0.1 to 1. We constructed a logical function $F\ (q,r)$ equal to 0 if (14) had imaginary roots, and 1 otherwise (Figure 7).

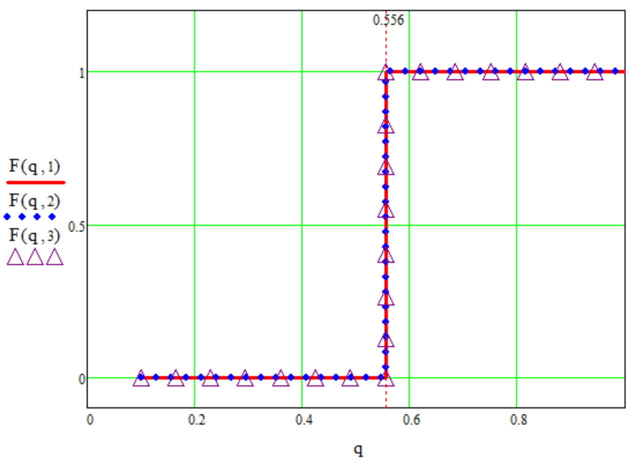


**Figure 7.** Graph of the logical function $F\ (q,r)$ for different values of $r$.

It followed from Figure 7 that the desired threshold value $q$ did not depend on the choice of $r$ and was $q_0 = 0.556$.

Thus, for $k > q_0 \cdot mg/r$, when the system under consideration is perturbed, periodic oscillations are guaranteed to occur.

*3.3. Mathematical Model of a Pendulum Taking into Account Friction Forces*

Let us introduce friction into the model (Figure 8).

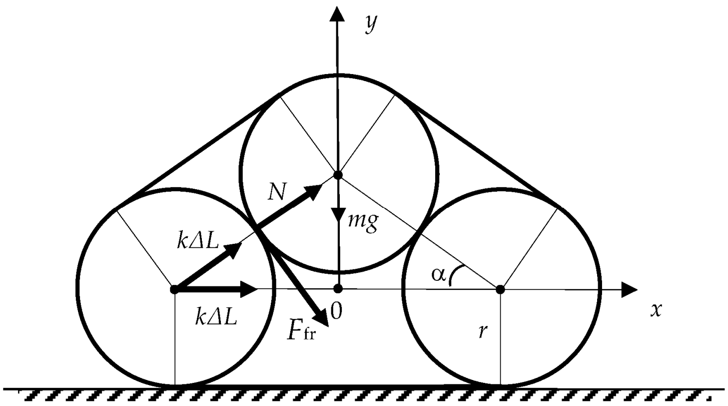


**Figure 8.** Mathematical model of a pendulum taking into account friction forces.

Let us construct the equations of motion of the cylinders (upper and lower left according to Figure 8), based on the balance of forces [34]:

$$\begin{aligned} my'' &= 2N\sin\alpha - mg - 2F_{\text{fr}}\cos\alpha \cdot \text{sign}y' - 2k\Delta L\sin\alpha; \\ mx'' &= k\Delta L(1+\cos\alpha) - F_{\text{fr}}\sin\alpha \cdot \text{sign}x' - N\cos\alpha, \end{aligned} \tag{15}$$

where $y$ $(t)$—top cylinder center's coordinate;

$x$ $(t)$—coordinate of the center of the lower left cylinder;

$N$—pressure force at the point of contact of the upper and lower cylinders (reaction force);

$F_{\text{fr}}$—sliding friction force between the upper and one lower cylinder (sliding friction of the thread on the cylinders and on the surface of the support was not taken into account).

Note that this model does not take into account the friction force between the cylinders and the surface, since this is not important from the point of view of a qualitative, rather than quantitative, study of the possibility of damped oscillations [35].

The system under consideration had one degree of freedom; therefore, instead of a system of two equations, we generated one equation of motion along the $Oy$ axis.

Express $x''$ in terms of the $y$ coordinate and its derivatives:

$$\frac{d^2}{dt^2}x = \frac{\partial}{\partial y}x \cdot \frac{d^2}{dt^2}y + \frac{\partial^2}{\partial y^2}x \cdot \left(\frac{d}{dt}y\right)^2 \tag{16}$$

where $x = -2r \cdot \cos\alpha$.

Given that:

$$\begin{aligned} \sin\alpha &= \tfrac{y}{2r}; \\ \cos\alpha &= \sqrt{1 - \tfrac{y^2}{(2r)^2}}, \end{aligned} \tag{17}$$

we obtain (16) in the form:

$$x'' = \frac{y}{\sqrt{(4r^2 - y^2)}} \cdot y'' + \frac{4r^2}{\sqrt{(4r^2 - y^2)^3}} \cdot (y')^2 \tag{18}$$

Note that $\mathrm{sign}y' = \mathrm{sign}x'$.

The friction force $F_{\mathrm{fr}}$ is defined as:

$$F_{\mathrm{fr}} = \mu N, \tag{19}$$

where $\mu$—coefficient of sliding friction between the upper and the lower cylinder.

From the second equation of system (15), taking into account (19), we obtain the reaction $N$:

$$N = \frac{k\Delta L(1+\cos\alpha) - mx''}{\mu \sin\alpha \cdot \mathrm{sign}y' + \cos\alpha}. \tag{20}$$

Substituting (17)–(20) into the first equation of system (15), we obtain the general differential equation of motion of the system, which, like Equation (9), is a nonlinear second-order equation.

$$y'' = \frac{\left(B_1 - 2k\left(\sqrt{4r^2-y^2} - r\right)\right)y - mgr - 2B_1\cos\alpha \cdot r \cdot \mu \cdot \mathrm{sign}y'}{mr - A_1 y + 2A_1\cos\alpha \cdot r \cdot \mu \cdot \mathrm{sign}y'} \tag{21}$$

where

$$\begin{gathered} A_1 = -\frac{2mry}{(4r^2-y^2) + \mu \cdot y \cdot \mathrm{sign}y' \cdot \sqrt{(4r^2-y^2)}}, \\ B_1 = \frac{2k\left(\sqrt{4r^2-y^2}-r\right)\left(2r+\sqrt{4r^2-y^2}\right)}{\sqrt{4r^2-y^2}+\mu \cdot y \cdot \mathrm{sign}y'} - \frac{8r^3 m\sqrt{4r^2-y^2}\cdot (y')^2}{\left(\sqrt{4r^2-y^2}+\mu\cdot y\cdot \mathrm{sign}y'\right)\cdot(4r^2-y^2)^2}. \end{gathered} \tag{22}$$

The result of the numerical solution of the differential Equation (21) in Mathcad, with the values $m$ = 1 kg, $r$ = 1 m, $k$ = 6.5 N/m, $\mu$ = 0.007 and the initial conditions $y(0) = r$, $y'(0) = 0$, is shown in Figure 9.

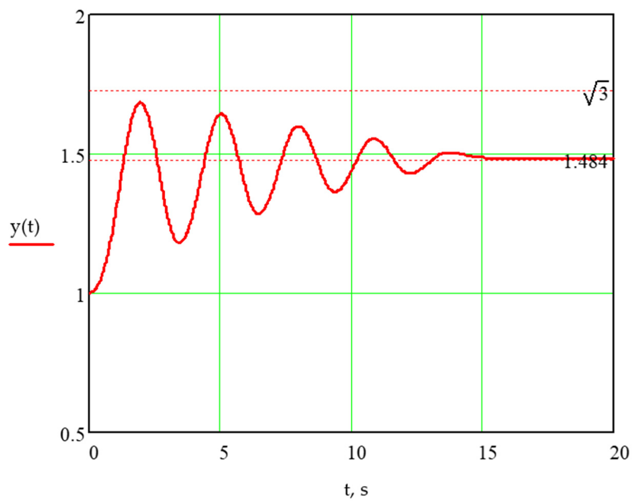


**Figure 9.** Solution of differential Equation (21).

Based on the graph in Figure 9, we observed damped periodic oscillations that the middle cylinder performed around the equilibrium position $y$ = 1.48 m (see Figure 3) when the system was perturbed, taking into account friction forces.

Note that Equation (21) with the friction coefficient $\mu$ = 0 is equivalent to Equation (9). Indeed, taking $\mu$ = 0 in Equation (21) and solving it with the same parameters as for (9), we obtained an oscillatory process that coincided with solution (9) to within 4 decimal places.

### *3.4. System Motion Study*

Let us analyze the motion of the system, having at our disposal the numerical trajectories of the motion of the middle cylinder, both with and without the forces of friction. As shown in Section 3.2, a necessary condition for the occurrence of oscillations is

$$k > q_0 \cdot \frac{mg}{r},\ q_0 = 0.556. \tag{23}$$

If condition (23) is satisfied, then there are three equilibrium positions of the upper cylinder in the system (for example, for the parameters $m = 1$ kg; $r = 1$ m; $q = 0.65$; $g = 9.8$):

$y_0 = 0$ m—stable resting position on the support surface;

$y_1 = 1.48$ m—stable position, in the vicinity of which oscillations can occur (point of minimum potential energy);

$y_2 = 0.86$ m —an unstable position from which the system moved to one of the stable positions (the point of maximum potential energy).

Figure 10 shows the motion of a frictionless system from various starting points $y$ (0):

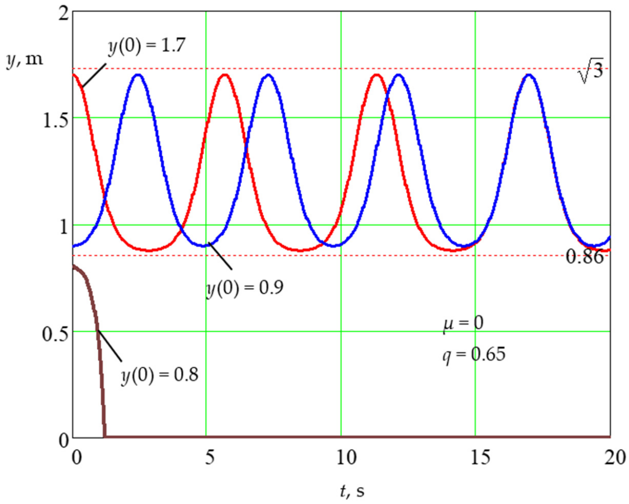


**Figure 10.** Movement of a system without friction from different starting points with a stiffness of the thread sufficient for the occurrence of free vibrations.

Figure 10 shows that oscillations were possible only if $y$ (0) > $y_2$ = 0.86 m.

Let us define the motion of the system with the stiffness of the thread $q \leq q_0$.

Figure 11 shows that with insufficient rigidity of the elastic thread, oscillations were impossible for any initial positions of the system. There was only one equilibrium position: $y_0 = 0$ m.

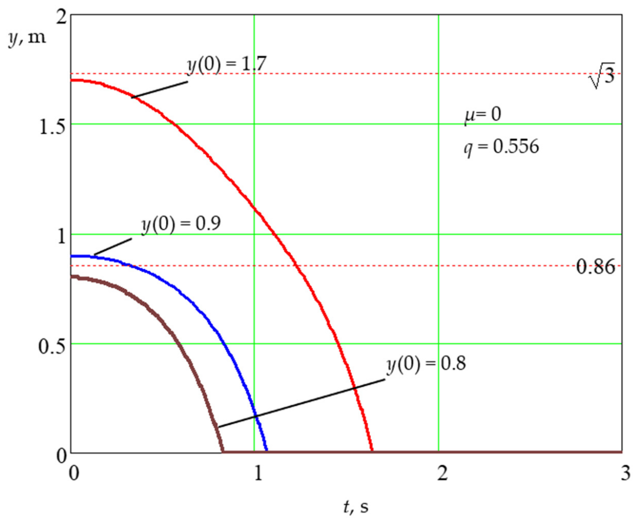


**Figure 11.** Movement of a frictionless system from different starting points with a rigidity of the thread insufficient for the occurrence of free vibrations.

Now let us consider similar situations, taking into account the forces of friction.

Motion with friction (Figure 12) differs from motion without friction (compare with Figure 10) by damping the oscillations. In addition, due to friction, the system could not start moving from the starting point $y$ (0) = 0.9 (see Figure 11), but only from a point with a larger value $y$ (0) = 0.925.

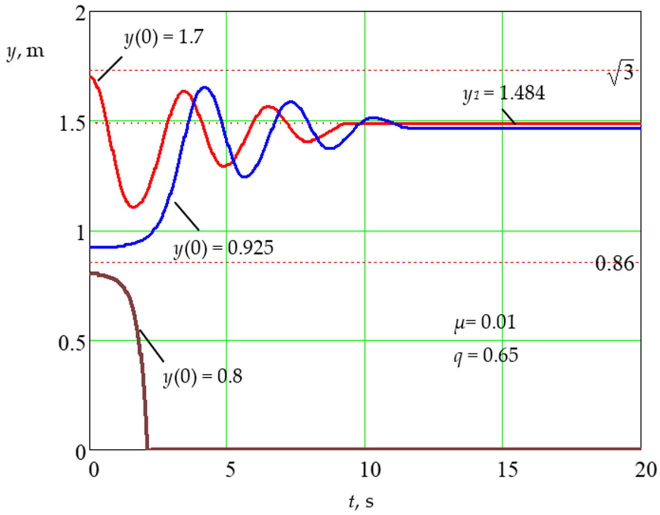


**Figure 12.** Motion of a system with friction from different starting points with a rigidity of the thread sufficient for the occurrence of free vibrations.

Movement with increased friction led to a rapid damping of the oscillations (see Figure 13).

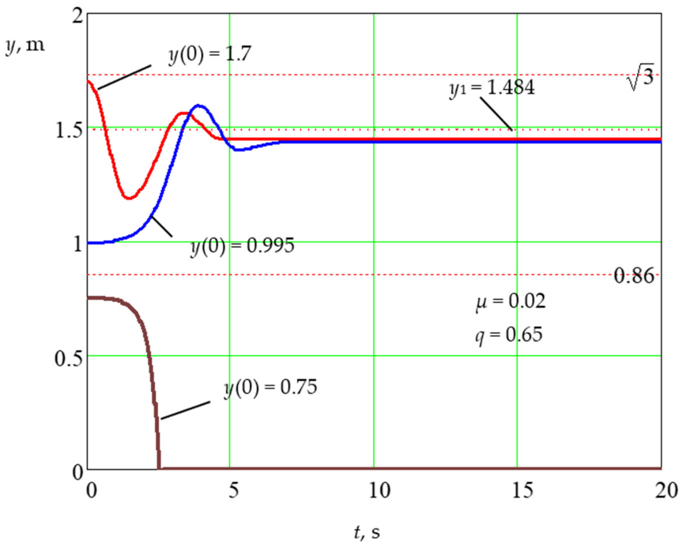


**Figure 13.** Movement of a system with an increased coefficient of friction from different initial points with a thread stiffness sufficient to cause free vibrations.

With a further increase in the coefficient of friction, starting approximately from $\mu = 0.04$, the movement of the system occurred without oscillations, as shown in Figure 14.

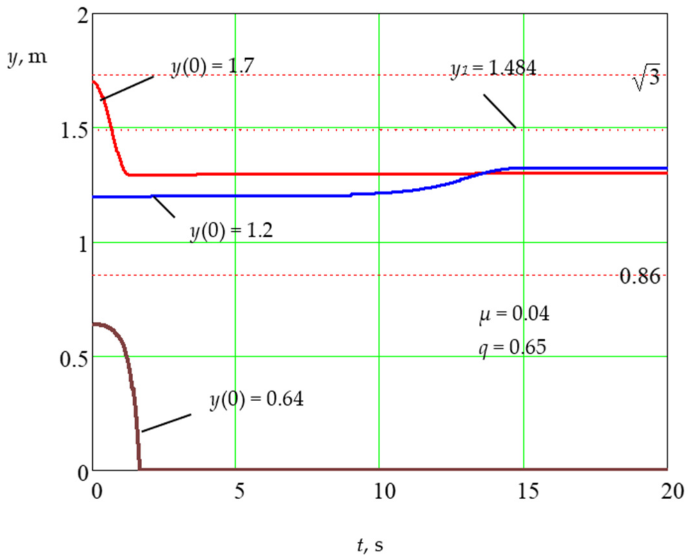


**Figure 14.** Movement of the system with a further increase in the coefficient of friction from different initial points with a stiffness of the thread sufficient for the occurrence of free vibrations.

## 4. Conclusions

In this work, the mathematical modeling of a nonlinear oscillatory system with elastic bonds, formed by three cylinders pulled together by an elastic thread, was carried out.

A technique for studying this system was developed, using the analysis of the potential energy of the system and the balance of forces acting on it.

As a result of the analysis of the potential energy of the system, it was shown that with a certain combination of parameters, the occurrence of periodic oscillations in the system was possible.

A method was proposed for determining the conditions for the occurrence of oscillations and the threshold value of the elasticity coefficient of an elastic thread, at which an oscillatory process is guaranteed to occur.

The technique was implemented in Mathcad by constructing and numerically solving the differential equation of motion of the system, both with and without taking into account the forces of friction between the cylinders.

The results obtained can be further extended to three-dimensional figures consisting of several elements in an elastic shell (for example, a three-tiered pyramid of 11 balls). In addition, it is possible to study a system with different element radii, taking into account the effect of heating on the stiffness coefficient, Hooke's nonlinear law, etc.

These results could be used as a basis for the analysis of complex nonlinear oscillatory systems with elastic bonds, in particular bioinspired mechatronic systems [36–39], as well as for educational purposes.

**Author Contributions:** Conceptualization, V.O.; methodology, V.O. and I.V.; software, Y.C. and A.T.; validation, V.O. and A.T.; formal analysis, I.V. and A.T.; investigation, I.V. and A.T.; resources, Y.C.; writing—original draft preparation, I.V.; writing—review and editing, V.O. and A.T.; visualization, V.O. and I.V.; project administration, V.O., Y.C. and A.T. All authors have read and agreed to the published version of the manuscript.

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**Data Availability Statement:** The datasets generated and analyzed during the current study are available from the corresponding author on reasonable request.

**Conflicts of Interest:** The authors declare no conflict of interest.

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