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From Transience to Recurrence for Cox–Ingersoll–Ross Model When $b < 0$

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Abstract: We consider the Cox–Ingersoll–Ross (CIR) model in time-dependent domains, that is, the CIR process in time-dependent domains reflected at the time-dependent boundary. This is a very meaningful question, as the CIR model is commonly used to describe interest rate models, and interest rates are often artificially set within a time-dependent domain by policy makers. We consider the most fundamental question of recurrence versus transience for normally reflected CIR process with time-dependent domains, and we examine some precise conditions for recurrence versus transience in terms of the growth rates of the boundary. The drift terms and the diffusion terms of the CIR processes in time-dependent domains are carefully provided. In the transience case, we also investigate the last passage time, while in the case of recurrence, we also consider the positive recurrence of the CIR processes in time-dependent domains.

Keywords: transience; recurrence; CIR model; time-dependent region; reflection

MSC: 60J60; 60K35; 60J80; 60J10

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1. Introduction

In mathematical finance, especially in the field of interest rate theory, the Cox–Ingersoll–Ross (CIR for short) model explains the evolution of interest rates. The CIR model is a type of one-factor model (short-rate model), as it describes interest rate movements as driven by only one source of market risk. The model was introduced by [1] as an extension of the Vasicek's interest rate model, and it has the following stochastic differential equation (SDE for short):

$$dX(t) = (a - bX(t))dt + \sigma\sqrt{X(t)}dW(t), \tag{1}$$

where $W(t)$ is a Wiener process (modeling the random market risk factor) and a, b, and σ are positive constants. The parameter a is the mean level or long-term interest rate constant, the parameter b is the speed of the mean reversion and corresponds to the speed of adjustment to the mean a, and σ regulates the volatility. The drift factor, $(a - bX(t))$, is the same as in the Vasicek model; see [2]. It ensures the mean reversion of the interest rate towards the long-run value a, with the speed of adjustment governed by the strictly positive parameter b. The stochastic volatility term $\sigma\sqrt{X(t)}dW(t)$ has a standard deviation that is proportional to the square root of the current rate. This implies that as the interest rate increases, its standard deviation increases, and as it falls and approaches zero, the stochastic volatility term also approaches 0.

In the following section, we mainly study the Equation (1) as the CIR model or CIR process. The same process is used in the Heston model, see [3], to model stochastic volatility. The SDE (1) has no explicit solution in general, even though its mean and variance can be calculated explicitly, and the probability transition density can be easily determined by using the time–space transformation. This CIR process $X(t)$ can be defined as a sum of squared Ornstein–Uhlenbeck process or be constructed using a BESQ process of dimension $d = \frac{4a}{\sigma^2}$; see [4]. Refs [5,6] proved that the CIR process is an affine process and the semigroup of every stochastic continuous affine process is a Feller semigroup; hence, the CIR process is a regular Feller process on the interval $(0, +\infty)$. The CIR process $X(t)$ is an ergodic process, and it possesses a stationary distribution. Furthermore, the CIR process is positive recurrent and nonexplosive on the interval $(0, +\infty)$.

Some diffusion processes in time-dependent domains have always been the focus of scholars' attention in the field of probability, and some various sample path properties involving diffusion processes in the time domain are constantly being discovered. The time-dependent domain problem that this article focuses on is actually a domain problem with deterministic moving boundaries, also known as noncylindrical domains. This type of time-dependent domain problem originates from both random environment problems and classic PDE problems with various boundary conditions; see [7]. In [8], the authors provided motivation for studying this issue of diffusion processes in time-dependent domains, through the theoretical explanation of a partial differential equation, and they focused on the heat equation in the time-dependent domain with Neumann rather than Dirichlet boundary conditions, that is, Brownian motion reflected on rather than killed at the boundary of a time-dependent domain. In [9], the most fundamental question of recurrence versus transience for normally reflected Brownian motion with time-dependent domains has been carefully studied, and the authors provided some sharp criterions for the recurrence versus transience of normally reflected Brownian motion in terms of the growth rate of the boundary. In [10] the author provided precise conditions for the recurrence versus transience of one-dimensional Brownian motion with a locally bounded drift, which belongs to the time-dependent domain with a normal reflection at the time-dependent boundary, and the precise conditions provided by the author naturally depend on the growth rates of the boundary and the drift terms of the diffusion processes.

Considering that the CIR model in time-dependent domains has important practical significance and value in the financial field, due to the fact that the evolution of interest rates is often limited to a regional scope, it often changes with the policies of interest rate makers or government management departments. In addition, this CIR model in time-dependent domains has theoretical significance in the field of mathematics and also promotes the research of the properties of transience versus recurrence for stochastic processes. Table 1 below gives the related progress in this field of transience versus recurrence for stochastic processes in time-dependent domains through the aid of the expression of the generator corresponding to the one-dimensional diffusion process. For more topics on the aspect of transience versus recurrence for stochastic processes, please refer to [11–14]. It should be pointed out that, in addition to transience versus recurrence for the conservative random walk, scaling limits for the conservative random walk have also been studied in the work of [11]. However, we did not address scaling limits for stochastic processes in this article.

Table 1. $L = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}$.

$\sigma^2(x)$	$b(x)$	Ref.
1	0	[8,9]
1	bx^γ	[10]
$\sigma^2 x$	$a - bx$	This paper

Here, we need to emphasize that in [8–10], they not only deal with one-dimensional situations, but also with multidimensional situations. For more detailed conclusions, please refer to the literature above for interested readers. In this paper, we only deal with the one-dimensional situations for technical reasons, but we deal with situations where $\sigma^2(x)$ is not a constant. At present, in this paper, we only deal with the case where $\sigma^2(x) = \sigma^2 x$ is linear, and of course, we can also consider the nonlinear case (which is not the CIR model). This problem will also be considered in a future work.

When $a \leq 0$, the CIR process hits zero repeatedly but after each hit becomes positive again; this behavior of hitting zero will also occur even if $a < \frac{1}{2}\sigma^2$. Therefore, we do not intend to handle this simple situation; we will only consider $a > 0$. At this point, the CIR process will have an upward positive constant slope a, and the evolution of the CIR process will still have a mean reversion property when $b > 0$. However, when we started considering $b < 0$, we saw that the CIR process will have a completely positive slope, which will encourage the CIR process to continuously move upwards and hit our constantly changing time-dependent upper boundary. If there are no time-dependent boundary restrictions, this will cause the CIR process to explode, thus possessing the property of transience. How is it possible to conduct the CIR process so as not to explode? In other words, how is it possible to transfer from transience to recurrence for the CIR process when $b < 0$? A natural idea is to add a boundary to the explosion diffusion process, just like the boundary of a time-dependent domain we mentioned above, and when this diffusion process hits the boundary, it will reflect back to our time-dependent domain. This is the main research topic of this paper, which is a fundamental problem in the field of probability, that is, recurrence versus transience, for this normally reflected CIR process with time-dependent domains.

In addition, in the transience case, we also investigate the last passage time, which plays an important and increasing role in financial modelling. The theory of the last passage time is a very important topic in the field of mathematical finance. In this paper, we only provide the probability distribution of the last passage time through the scale function, without exploring its application in the financial modelling field. See [15], as well as [16], for the applications the last passage time to hazard processes and models of default risk.

Let us briefly explain the analytical method we used to prove recurrence versus transience for this normally reflected CIR process with time-dependent domains. The first major tool is the well-known Feynman–Kac formula of diffusion process, which provides the stochastic representation for the solution to the boundary value problem. It is worth noting here that the common Feynman–Kac formula is a boundary value problem with a Dirichlet condition or Cauchy condition; however, we still need the Feynman–Kac formula for the boundary value problem with a Neumann boundary condition here, as we need to handle the normally reflected CIR process with time-dependent domains. The second tool we use is the criticality theory of second-order elliptic operators; in particular, the maximum principle or comparison theorem is frequently used in our proofs. It is worth mentioning that some comparison theorems are not clearly found in the literature, and we provide detailed proofs of them in the Appendix. Regarding the criticality theory, we refer the reader to [17] for more details. Due to the need to obtain precise conditions for coefficients in the CIR process, the selection of certain parameters is also crucial in our proof process.

This paper is structured as follows. In Section 2, we give some basic notations used throughout this paper and provide some auxiliary results about the moment generating function of the first hitting time. In Section 3, we prove the results of two recurrent properties, recurrence and positive recurrence, and provide the precise conditions that the coefficients of the CIR process should meet for recurrence and positive recurrence in terms of the growth rates of the boundary, the drift terms, and the diffusion terms of the CIR processes in time-dependent domains. In Section 4, we prove the result of the transient for the CIR process in time-dependent domains and also provide the precise conditions that the coefficients of the CIR process should meet. Section 5 concludes, and in Appendix A, we provide some comparison theorems of second-order ordinary differential equations with nonconstant coefficients. in Appendix B, the exact solution of a second-order ordinary differential equation with nonconstant coefficients is given by transforming it into one-dimensional Riccati equation.

2. Auxiliary Results

We will first introduce some notations, which we will frequently use in the following sections. Let $X(t)$ denote a canonical, continuous, real-valued path, and let $T_\alpha = \inf\{t \geq 0 : X(t) = \alpha\}$. We introduce some generators for some diffusion processes:

$$\begin{aligned} L_{bx^\gamma} &= \frac{1}{2}\frac{d^2}{dx^2} + bx^\gamma \frac{d}{dx}; \\ L_D &= \frac{1}{2}\frac{d^2}{dx^2} + D\frac{d}{dx}; \\ L_{CIR} &= \frac{1}{2}\sigma^2 x \frac{d^2}{dx^2} + (a - bx)\frac{d}{dx}. \end{aligned}$$

Let $P_x^{*;Ref\leftarrow:\beta}$ and $E_x^{*;Ref\leftarrow:\beta}$ denote probabilities and expectations for diffusion process corresponding to the generator L_* on $[1,\beta]$, starting from $x \in [1,\beta]$, with a reflection at β and stopped at 1. Let $P_x^{*;Ref\rightarrow:\alpha}$ and $E_x^{*;Ref\rightarrow:\alpha}$ denote the probabilities and expectations for diffusion process corresponding to the generator L_* on $[\alpha,\infty)$, starting from $x \in [\alpha,\infty)$, with a reflection at α.

2.1. Moment-Generating Functions

Next, we will provide some auxiliary results about the moment-generating function of the first hitting time using some diffusion process without proofs. Actually, these conclusions can be easily obtained from the well-known Feynman–Kac formula and the criticality theory of second-order elliptic operators after simple calculations.

(A) It follows from the Feynman–Kac formula that

$$u(x) = E_x^{D;Ref\rightarrow:1}[e^{\frac{D^2}{2}T_\beta}]$$

solves the boundary value problem

$$\begin{cases} (L_D + \frac{D^2}{2})u = 0, & \text{in } (1,\beta), \\ u'(1) = 0, \\ u(\beta) = 1. \end{cases}$$

The solution of this linear equation is given by the function

$$u(x) = \frac{1}{1 + D(\beta - 1)}(1 + D(x-1))e^{D(\beta-1)}.$$

According to the criticality theory of second-order elliptic operators, for instance, see [17], it follows that the principal eigenvalue λ_1 for $-L_D$ satisfies

$$\lambda_1(-L_D) \geq \frac{D^2}{2}.$$

(B) It follows from the Feynman–Kac formula that

$$u(x) = E_x^{D;Ref\leftarrow:\beta}[e^{-\lambda T_\alpha}]$$

solves the boundary value problem, $\lambda > 0$:

$$\begin{cases} (L_D - \lambda)u = 0, & \text{in } (\alpha,\beta), \\ u(\alpha) = 1, \\ u'(\beta) = 0 \end{cases}$$

The solution of this linear equation is given by the function

$$u(x) = \frac{r_1 e^{r_2 x - r_1 \beta} + r_2 e^{-r_1 x + r_2 \beta)}}{r_1 e^{r_2 \alpha - r_1 \beta} + r_2 e^{-r_1 \alpha + r_2 \beta)}},$$

where $r_1 = D + \sqrt{D^2 + 2\lambda}, r_2 = -D + \sqrt{D^2 + 2\lambda}$.

(C) It follows from the Feynman–Kac formula that

$$u(x) = E_x^{D;Ref \to :1}[e^{DT_\beta}]$$

solves the boundary value problem

$$\begin{cases} (L_D + D)u = 0, & \text{in } (1, \beta), \\ u'(1) = 0, \\ u(\beta) = 1. \end{cases}$$

The solution of this linear equation is given by the function

$$u(x) = \frac{r_1 e^{-(r_2 x + r_1)} - r_2 e^{-(r_1 x + r_2)}}{r_1 e^{-(r_2 \beta + r_1)} - r_2 e^{-(r_1 \beta + r_2)}},$$

where $r_1 = D - \sqrt{D^2 - 2D}, r_2 = D + \sqrt{D^2 - 2D}$.

(D) It follows from the Feynman–Kac formula that

$$u(x) = E_x^{D;Ref \to :1}[e^{\lambda T_\beta}]$$

solves the boundary value problem

$$\begin{cases} (L_D + \lambda)u = 0, & \text{in } (1, \beta), \\ u'(1) = 0, \\ u(\beta) = 1. \end{cases}$$

The solution of this linear equation is given by the function

$$u(x) = \frac{r_1 e^{-r_2 x - r_1} - r_2 e^{-r_1 x - r_2}}{r_1 e^{-r_2 \beta - r_1} - r_2 e^{-r_1 \beta - r_2}},$$

where $r_1 = D - \sqrt{D^2 - 2\lambda}, r_2 = D + \sqrt{D^2 - 2\lambda}$.

2.2. Moment-Generating Function for CIR Model

Consider the following CIR model:

$$dX_t = (a - bX_t)dt + \sigma\sqrt{X_t}dW_t,$$

with its operator

$$L_{CIR} = \frac{1}{2}\sigma^2 x \frac{d^2}{dx^2} + (a - bx)\frac{d}{dx}.$$

Lemma 1. *(i) The function*

$$u_\lambda(x) := E_x^{CIR;Ref \leftarrow :\beta} e^{\lambda T_\alpha}$$

satisfies the following boundary value problem

$$\begin{cases} (L_{CIR} + \lambda)u = 0, & \text{in } (\alpha, \beta), \\ u(\alpha) = 1, \\ u'(\beta) = 0. \end{cases}$$

(ii) *For $\alpha \in [1, \beta]$ and $\lambda \leq \hat{\lambda}(\alpha, \beta)$,*

$$E_x^{CIR;Ref\leftarrow:\beta} e^{\lambda T_\alpha} \leq 2,$$

where $x \in [\alpha, \beta]$, and

$$\hat{\lambda}(\alpha, \beta) = -\frac{b}{\sigma^2} e^{-\frac{2(1+a-b\alpha)}{\alpha\sigma^2}(\beta-\alpha)} = -\frac{b}{\sigma^2} e^{(\frac{2b}{\sigma^2} - \frac{2(1+a)}{\alpha\sigma^2})(\beta-\alpha)}.$$

Proof. It is easy to obtain (i) from the well-known Feynman–Kac formula, and we will only prove that (ii) holds. Consider the function

$$u(x) = 2 - e^{-r(x-\alpha)}, \ \ \alpha \leq x \leq \beta,$$

where $r > 0$. Choose $-\frac{1}{2}\sigma^2 r < b < 0$. Then,

$$\begin{aligned} e^{r(x-\alpha)}(L_{CIR} + \lambda)u &= -r(b + \frac{1}{2}\sigma^2 r)x + ar - \lambda + 2\lambda e^{r(x-\alpha)} \\ &\leq -r(b + \frac{1}{2}\sigma^2 r)\alpha + ar - \lambda + 2\lambda e^{r(\beta-\alpha)} \\ &= -\frac{1}{2}\alpha\sigma^2 r^2 + (a - b\alpha)r + \lambda(2e^{r(\beta-\alpha)} - 1). \end{aligned}$$

Next, we solve the following inequality,

$$\begin{aligned} &-\frac{1}{2}\alpha\sigma^2 r^2 + (a - b\alpha)r + \lambda(2e^{r(\beta-\alpha)} - 1) \leq 0 \\ \text{and} \quad &\lambda(2e^{r(\beta-\alpha)} - 1) \leq \frac{1}{2}\alpha\sigma^2 r^2 - (a - b\alpha)r = (\frac{1}{2}\alpha\sigma^2 r - (a - b\alpha))r, \end{aligned}$$

so that we obtain

$$\lambda \leq \frac{(\frac{1}{2}\alpha\sigma^2 r - (a - b\alpha))r}{2e^{r(\beta-\alpha)} - 1}.$$

Hence, we have

$$(L_{CIR} + \lambda)u \leq 0, \ \ \text{in } (\alpha, \beta),$$

if

$$0 \leq \lambda \leq \frac{(\frac{1}{2}\alpha\sigma^2 r - (a - b\alpha))r}{2e^{r(\beta-\alpha)} - 1}.$$

Let

$$\frac{1}{2}\alpha\sigma^2 r - (a - b\alpha) = 1,$$

and we can choose

$$r = \frac{2(1 + a - b\alpha)}{\alpha\sigma^2} = -\frac{2b}{\sigma^2} + \frac{2(1 + a)}{\alpha\sigma^2}.$$

Obviously, r satisfies $r > -\frac{2b}{\sigma^2}$. After tedious calculations,

$$\begin{aligned} &\frac{(\frac{1}{2}\alpha\sigma^2 r - (a - b\alpha))r}{2e^{r(\beta-\alpha)} - 1} \\ = \ &\frac{r}{2e^{r(\beta-\alpha)} - 1} = \frac{\frac{2(1+a-b\alpha)}{\alpha\sigma^2}}{2e^{r(\beta-\alpha)} - 1} \\ \geq \ &\frac{\frac{2(1+a-b\alpha)}{\alpha\sigma^2}}{2e^{r(\beta-\alpha)}} \geq \frac{\frac{-b\alpha}{\alpha\sigma^2}}{e^{r(\beta-\alpha)}} = \frac{-b}{\sigma^2} e^{-r(\beta-\alpha)}, \end{aligned}$$

and choosing

$$r = \frac{2(1 + a - b\alpha)}{\alpha\sigma^2},$$

we can obtain

$$\frac{(\frac{1}{2}\alpha\sigma^2 r - (a - b\alpha))r}{2e^{r(\beta-\alpha)} - 1} \geq -\frac{b}{\sigma^2} e^{-\frac{2(1+a-b\alpha)}{\alpha\sigma^2}(\beta-\alpha)} := \hat{\lambda}(\alpha, \beta),$$

Here, we provide the definition of $\hat{\lambda}(\alpha, \beta)$, which we will frequently use below.

We have thus shown that there exists a function $u > 0$ on $[\alpha, \beta]$ satisfying

$$\begin{cases} (L_{CIR} + \hat{\lambda})u \leq 0, & \text{in } (\alpha, \beta), \\ u(\alpha) = 1, \\ u'(\beta) \geq 0. \end{cases}$$

Let $\lambda_1(-L)$ be the principal eigenvalue of the second-order elliptic operator for $-L$; according to the criticality theory of the second-order elliptic operators, it follows that the principal eigenvalue $\lambda_1(-L_{CIR})$ satisfies

$$\lambda_1(-L_{CIR}) \geq \hat{\lambda},$$

where the second-order elliptic operator L_{CIR} satisfies

$$\begin{cases} L_{CIR} u = 0, & \text{in } (\alpha, \beta), \\ u(\alpha) = 1, \\ u'(\beta) = 0. \end{cases}$$

According to the Feynman–Kac formula, if $\lambda \leq \lambda_1(-L_{CIR})$, then the function

$$u_\lambda(x) := E_x^{CIR;Ref\leftarrow:\beta} e^{\lambda T_\alpha}$$

satisfies the following boundary value problem

$$\begin{cases} (L_{CIR} + \lambda)u = 0, & \text{in } (\alpha, \beta), \\ u(\alpha) = 1, \\ u'(\beta) = 0. \end{cases}$$

According to the generalized maximum principal, it follows from $\lambda \leq \lambda_1(-L_{CIR})$ that

$$u_\lambda \leq u,$$

where, u satisfies

$$\begin{cases} (L_{CIR} + \lambda)u \leq 0, & \text{in } (\alpha, \beta), \\ u(\alpha) = 1, \\ u'(\beta) \geq 0, \end{cases} \tag{2}$$

Obviously, $u(x) = 2 - e^{-r(x-\alpha)}$ satisfies (2). Hence, in particular, we have

$$E_x^{CIR;Ref\leftarrow:\beta} e^{\lambda T_\alpha} = u_\lambda(x) \leq u(x) \leq 2.$$

This completes the proof of this lemma. □

3. Recurrence of the CIR Model When $b < 0$

Transience recurrence dichotomous issues are central to the study of stochastic processes and help describe the stochastic process's overall structure. There are many equivalent definitions of transience versus recurrence dichotomy in many of the literature; here, we can refer to [18,19].

Definition 1. *The stochastic process $X(t)$ is recurrent if $X(t)$ belongs to $\mathcal{O}$ at arbitrarily large times t, with a probability of one, and is transient if $X(t)$ belongs to $\mathcal{O}$ at arbitrarily large times t, with a probability of zero, for any set $\mathcal{O}$.*

In this section, we prove the results of two recurrent properties, that is, recurrence and positive recurrence. The definition of positive recurrence for a stochastic process $X(t)$ is given in the following subsection.

3.1. Recurrence

Theorem 1. *Consider the CIR model corresponding to the generator*

$$\frac{1}{2}\sigma^2 x\frac{d^2}{dx^2} + (a - bx)\frac{d}{dx}$$

in the time-dependent region $[1, f(t)]$ with reflection at both the fixed endpoint and the time-dependent one. Let $\sigma > 0, b < 0$, and $a > 0$ satisfy that there is an $a_0 < \frac{1}{2}$ and an $a \leq a_0$, where a_0 solves $2a = e^{\frac{2b}{\sigma^2} - \frac{2(1+a)}{\sigma^2}}$. Assume that $f(t) \leq \ln t$ for sufficiently large t. If

$$b > -\frac{\sigma^2}{2},$$

or if

$$b = -\frac{\sigma^2}{2} \text{ and } a < \frac{\sigma^2}{2},$$

then the CIR model is recurrent.

Proof. Let $j_0 \geq 3$ and $t_j = e^j$. Then, we have $f(t_j) > 2$ for $j \geq j_0$. For $j \geq j_0$, let A_{j+1} denote the event that the CIR process hits 1 at some time $t \in [t_j, t_{j+1}]$. The conditional version of the Borel–Cantelli lemma shows that if

$$\sum_{j=j_0}^{\infty} P_1(A_{j+1}|\mathcal{F}_{t_j}) = \infty, \quad \text{a.s.}, \tag{3}$$

then $P_1(A_j, \text{i.o.}) = 1$, and thus the CIR process is recurrent. Thus, to show recurrence of the CIR process, it suffices to show (3).

Since up to time t_j, the largest that the CIR process can be is $f(t_j)$, and since up to time t_{j+1}, the time-dependent region is contained in $[1, f(t_{j+1})]$, it follows by comparison that

$$P_1(A_{j+1}|\mathcal{F}_{t_j}) \geq P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}(T_1 \leq t_{j+1} - t_j), \quad \text{a.s.} \tag{4}$$

Now, we estimate $P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}(T_1 \leq t_{j+1} - t_j)$. Let

$$\begin{aligned}
\sigma_0^{(j)} &= 0, \\
\tau_i^{(j)} &= \inf\{t \geq \sigma_{i-1}^j | X(t) = f(t_{j+1})\}, \\
\sigma_i^{(j)} &= \inf\{t > \tau_i^{(j)} | X(t) = f(t_j)\}, \quad j \geq j_0, \ \ i = 1, 2, \cdots.
\end{aligned}$$

For any $l_j \in \mathbb{N}$,

$$\{T_1 < \sigma_{l_j}^{(j)}\} - \{\sigma_{l_j}^{(j)}\} > t_{j+1} - t_j\} \subset \{T_1 < t_{j+1} - t_j\}.$$

It follows from the strong Markov property that

$$P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}(T_1 < \sigma_{l_j}^{(j)}) = 1 - \left(P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}(T_{f(t_{j+1})} < T_1)\right)^{l_j}.$$

Thus, we have

$$\begin{aligned} & P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}(T_1 \le t_{j+1} - t_j) \\ \ge\ & 1 - \left(P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}(T_{f(t_{j+1})} < T_1)\right)^{l_j} - P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}(\sigma_{l_j}^{(j)} > t_{j+1} - t_j). \end{aligned}$$

We then will obtain $P_1(A_j \text{ i.o.}) = 1$, and thus recurrence, if we can select $\{l_j\}_{j=1}^{\infty}$ such that

$$\sum_{j=j_0}^{\infty} \left(1 - \left(P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}(T_{f(t_{j+1})} < T_1)\right)^{l_j}\right) = \infty, \tag{5}$$

and

$$\sum_{j=j_0}^{\infty} P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}(\sigma_{l_j}^{(j)} > t_{j+1} - t_j) < \infty. \tag{6}$$

Define

$$\phi(x) := \int_x^{\infty} t^{-\frac{2a}{\sigma^2}} e^{\frac{2b}{\sigma^2}(t-1)} dt.$$

Obviously,

$$L^{CIR}\phi(x) = 0.$$

According to the standard probabilistic potential theory, it follows that

$$\begin{aligned} & P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}(T_{f(t_{j+1})} < T_1) \\ =\ & \frac{\phi(1) - \phi(f(t_j))}{\phi(1) - \phi(f(t_{j+1}))} \\ =\ & 1 - \frac{\phi(f(t_j)) - \phi(f(t_{j+1}))}{\phi(1) - \phi(f(t_{j+1}))}. \end{aligned}$$

We compute using L'Hôpital's rule that

$$\lim_{x\to\infty} \frac{\phi(x)}{-\phi'(x)} = -\frac{\sigma^2}{2b},$$

and, we get, as $x \to \infty$,

$$\phi(x) \sim \frac{\sigma^2}{2b}\phi'(x) = -\frac{\sigma^2}{2b} x^{-\frac{2a}{\sigma^2}} e^{\frac{2b}{\sigma^2}(x-1)},$$

where $\sim$ indicates asymptotic equality in the sense that the ratio of the two sides goes to 1 as $x \to \infty$. Using the fact that

$$(1-t)^l \le e^{-lt} \le 1 - lt + \frac{1}{2}(lt)^2 \le 1 - \frac{1}{2}lt,$$

if $l, t \ge 0$ and $lt \le 1$, we have

$$1 - \left(P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}(T_{f(t_{j+1})} < T_1)\right)^{l_j} \ge \frac{1}{2} l_j \frac{\phi(f(t_j)) - \phi(f(t_{j+1}))}{\phi(1) - \phi(f(t_{j+1}))}, \tag{7}$$

for sufficiently large j, if $\lim_{j\to\infty} l_j\phi(f(t_j)) = 0$. Obviously, we can choose a $C_0 \in (0,1)$ such that $\phi(f(t_{j+1})) \le C_0\phi(f(t_j))$ for all large j. Thus, for all sufficiently large j, we have

$$\frac{\phi(f(t_j)) - \phi(f(t_{j+1}))}{\phi(1) - \phi(f(t_{j+1}))} \ge C_1\phi(f(t_j)) \ge C_2 j^{-\frac{2a}{\sigma^2}} e^{\frac{2b}{\sigma^2}(j-1)},$$

for some constants $C_1, C_2 > 0$. Now, we choose $l_j \in \mathbb{N}$ according to

$$l_j := \Big[\frac{1}{\log j} j^{\frac{2a}{\sigma^2}-1} e^{-\frac{2b}{\sigma^2}(j-1)}\Big].$$

Hence, we obtain

$$\begin{aligned} &\sum_{j=j_0}^{\infty}\left(1 - \left(P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}(T_{f(t_{j+1})} < T_1)\right)^{l_j}\right) \\ \ge\ & \sum_{j=j_0}^{\infty} C l_j j^{-\frac{2a}{\sigma^2}} e^{\frac{2b}{\sigma^2}(j-1)} \\ \ge\ & \sum_{j=j_0}^{\infty} C\frac{1}{j\log j} \\ =\ & \infty, \end{aligned}$$

for the constant $C > 0$.

With l_j chosen as above, we now analyze the second term

$$P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}(\sigma_{l_j}^{(j)} > t_{j+1} - t_j).$$

It follows from the strong Markov property that

$$\sigma_{l_j}^{(j)} = \sum_{i=1}^{l_j} X_i + \sum_{i=1}^{l_j} Y_i,$$

where $\{X_i\}_{i=1}^{\infty}$ is an independent and identically distributed sequence distributed according to $T_{f(t_{j+1})}$ under $P_{f(t_j)}^{CIR;Ref\rightarrow:1}$, $\{Y_i\}_{i=1}^{\infty}$ is an independent and identically distributed sequence distributed according to $T_{f(t_j)}$ under $P_{f(t_{j+1})}^{CIR;Ref\leftarrow:f(t_{j+1})}$, and the two sequences are independent of one another.

For any $\lambda > 0$, according to Markov's inequality,

$$\begin{aligned} & P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}(\sigma_{l_j}^{(j)} > t) \\ \le\ & e^{-\lambda t} E_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}[e^{\lambda\sigma_{l_j}^{(j)}}] \\ =\ & e^{-\lambda t} E_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}[e^{\lambda\sum_{i=1}^{l_j} X_i} e^{\lambda\sum_{i=1}^{l_j} Y_i}] \\ =\ & e^{-\lambda t}\left(E_{f(t_j)}^{CIR;Ref\rightarrow:1}[e^{\lambda T_{f(t_{j+1})}}]\right)^{l_j}\left(E_{f(t_{j+1})}^{CIR;Ref\leftarrow:f(t_{j+1})}[e^{\lambda T_{f(t_j)}}]\right)^{l_j}. \end{aligned} \tag{8}$$

According to Lemma 1,

$$E_{f(t_{j+1})}^{CIR;Ref\leftarrow:f(t_{j+1})}[e^{\lambda T_{f(t_j)}}] \le 2, \tag{9}$$

for $\lambda \leq \hat{\lambda}(f(t_j), f(t_{j+1}))$, where $\hat{\lambda}(\cdot,\cdot)$ is as in Lemma 1. Using the fact that $f(t_j) = \ln(e^j) = j$, it is easy to check that there exists a $\hat{\lambda}_0 > 0$ such that $\hat{\lambda}(f(t_j), f(t_{j+1})) \geq \hat{\lambda}_0$ for all $j \geq 1$. In fact, by the definition of $\hat{\lambda}(\cdot,\cdot)$ in Lemma 1,

$$\hat{\lambda}(f(t_j), f(t_{j+1})) = -\frac{b}{\sigma^2} e^{\frac{2b}{\sigma^2} - \frac{2(1+a)}{\sigma^2 j}} \to -\frac{b}{\sigma^2} e^{\frac{2b}{\sigma^2}}$$

as $j \to \infty$ if $1 + a > 0$. Hence, we obtain

$$\hat{\lambda}_0 = -\frac{b}{\sigma^2} e^{\frac{2b}{\sigma^2} - \frac{2(1+a)}{\sigma^2}} > 0.$$

By choosing $\lambda_j = -\frac{2ab}{\sigma^2 j}$, there exists a j_0, and we have $\lambda_j \leq \hat{\lambda}_0$ for all $j \geq j_0$. By choosing $\lambda = \lambda_1 = -\frac{2ab}{\sigma^2}$, we have $\lambda \leq \hat{\lambda}_0$ if a satisfies the following inequality:

$$a \leq \frac{1}{2} e^{\frac{2b}{\sigma^2} - \frac{2(1+a)}{\sigma^2}},$$

that is, there is a $a_0 < \frac{1}{2}$, $a \leq a_0$, where a_0 solves $2a = e^{\frac{2b}{\sigma^2} - \frac{2(1+a)}{\sigma^2}}$.

Using Lemma A6, we substitute $x = f(t_j) = j$ and $\beta = f(t_{j+1}) = (j+1)$ in the expression on the right-hand side of (A11); the resulting expression is bounded in j. In fact, it follows from (A11) that

$$\begin{aligned}
u(j) &= e^{-\frac{2b}{\sigma^2 j}} \exp\left(- \int_j^{j+1} \frac{\exp(-\frac{2b(y-1)}{\sigma^2 j}) y^{-\frac{2a}{\sigma^2}}}{\int_1^y \exp(-\frac{2b(t-1)}{\sigma^2 j}) t^{-\frac{2a}{\sigma^2}} dt - \frac{\sigma^2 j}{2b}} dy \right) \\
&\leq e^{-\frac{2b}{\sigma^2 j}} \exp\left(- \int_j^{j+1} \frac{\exp(-\frac{2b(j-1)}{\sigma^2 j}) (j+1)^{-\frac{2a}{\sigma^2}}}{\int_1^{j+1} \exp(-\frac{2b(t-1)}{\sigma^2 j}) t^{-\frac{2a}{\sigma^2}} dt - \frac{\sigma^2 j}{2b}} dy \right) \\
&\leq e^{-\frac{2b}{\sigma^2 j}} \exp\left(- \frac{\exp(-\frac{2b(j-1)}{\sigma^2 j}) (j+1)^{-\frac{2a}{\sigma^2}}}{\int_1^{j+1} \exp(-\frac{2bj}{\sigma^2 j}) dt - \frac{\sigma^2 j}{2b}} \right) \\
&= e^{-\frac{2b}{\sigma^2 j}} \exp\left(- \frac{\exp(-\frac{2b(j-1)}{\sigma^2 j}) (j+1)^{-\frac{2a}{\sigma^2}}}{\int_1^{j+1} \exp(-\frac{2b}{\sigma^2}) dt - \frac{\sigma^2 j}{2b}} \right) \\
&\leq e^{-\frac{2b}{\sigma^2 j}} \exp\left(- \frac{\exp(-\frac{2b(j-1)}{\sigma^2 j}) (j+1)^{-\frac{2a}{\sigma^2}}}{(j+1)\exp(-\frac{2b}{\sigma^2}) - \frac{\sigma^2}{2b}(j+1)} \right) \\
&= e^{-\frac{2b}{\sigma^2 j}} \exp\left(- \frac{1}{(j+1)^{1+\frac{2a}{\sigma^2}} \exp(-\frac{2b}{\sigma^2 j}) - \frac{\sigma^2}{2b} \exp(\frac{2b(j-1)}{\sigma^2 j})(j+1)^{1+\frac{2a}{\sigma^2}}} \right).
\end{aligned}$$

Obviously, notice that when $b < 0$,

$$\lim_{j\to\infty} \frac{1}{(j+1)^{1+\frac{2a}{\sigma^2}} \exp(-\frac{2b}{\sigma^2 j}) - \frac{\sigma^2}{2b} \exp(\frac{2b(j-1)}{\sigma^2 j})(j+1)^{1+\frac{2a}{\sigma^2}}} = 0.$$

Hence, we have

$$u(j) \leq e^{-\frac{2b}{\sigma^2 j}} \leq e^{-\frac{2b}{\sigma^2}},$$

for sufficiently large $j \geq 1$ when $b < 0$.

By letting $M := e^{-\frac{2b}{\sigma^2}} > 1$ be an upper bound, it follows that

$$E_{f(t_j)}^{CIR;Ref\to:1} e^{\lambda T_{f(t_{j+1})}} = E_{f(t_j)}^{CIR;Ref\to:1} e^{-\frac{2ab}{\sigma^2} T_{f(t_{j+1})}} \le M. \tag{10}$$

By noting that $t_{j+1} - t_j = e^{j+1} - e^j \ge e^j$, it follows from (8) that

$$P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})} (\sigma_{l_j}^{(j)} > t_{j+1} - t_j) \le e^{\frac{2ab}{\sigma^2} e^j} (2M)^{l_j}, \tag{11}$$

for sufficiently large j. Recalling the expression of l_j, we can have

$$\begin{aligned} & P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})} (\sigma_{l_j}^{(j)} > t_{j+1} - t_j) \\ \le\ & e^{\frac{2ab}{\sigma^2} e^j} (2M)^{\frac{1}{\log j} j^{\frac{2a}{\sigma^2} - 1} e^{-\frac{2b}{\sigma^2}(j-1)}} \\ =\ & e^{\frac{2ab}{\sigma^2} e^j} e^{\frac{1}{\log j} j^{\frac{2a}{\sigma^2} - 1} e^{-\frac{2b}{\sigma^2}(j-1)} \log 2M}, \end{aligned} \tag{12}$$

for sufficiently large j. It follows that the right-hand side of (12) is summable in j if $1 > -\frac{2b}{\sigma^2}$, that is,

$$b > -\frac{\sigma^2}{2},$$

or if

$$b = -\frac{\sigma^2}{2} \text{ and } a \le \frac{\sigma^2}{2}.$$

Thus, (6) holds for this range of a, b and σ. This completes the proof of this theorem. □

Remark 1. *In the time-independent region case, it is known that the drift $a - bX_t$ ensures a mean reversion of the CIR model towards the long-term value $\frac{a}{b}$. In the time-dependent region case, however, the CIR model can reflect at the fixed endpoint 1. Obviously, $a \le \frac{1}{2} e^{\frac{2b}{\sigma^2}} < \frac{1}{2}$. This guarantees that the CIR model can down-cross the boundary 1; hence, the CIR model can reflect at the fixed point 1 infinitely often.*

3.2. Positive Recurrence

Now that we have the recurrence of the CIR model, it is natural to consider the positive recurrence in the following sense. The following definition of positive recurrence for a stochastic process can be found in [18].

Definition 2. *We say that a one-dimensional process is a positive recurrence if, starting from $x > 1$, the expected value of the first hitting time of 1 is finite, that is,*

$$E_x T_1 < \infty.$$

Theorem 2. *Consider the CIR model corresponding to the generator*

$$\frac{1}{2} \sigma^2 x \frac{d^2}{dx^2} + (a - bx) \frac{d}{dx}$$

in the time-dependent region $[1, f(t)]$, with reflection at both the fixed endpoint 1 and the time-dependent endpoint $f(t)$ at time t. Let $\sigma > 0, b < 0$, and $a > 0$ satisfy that there is an $a_0 < \frac{1}{2}$ and an $a \le a_0$, where a_0 solves $2a = e^{\frac{2b}{\sigma^2} - \frac{2(1+a)}{\sigma^2}}$. Assume that $f(t) \le \ln t$, for sufficiently large t. If

$$b > -\frac{\sigma^2}{2},$$

then the CIR model is positive recurrent.

Proof. Let P_2 and E_2 denote probabilities and expectations for the process starting from $x = 2$ at time 0. Let $t_j = e^j$ as in the proof of Theorem 1. We have

$$E_2T_1 \le t_1 + \sum_{j=1}^{\infty} t_{j+1}P_2(T_1 \ge t_j) = e + \sum_{j=1}^{\infty} e^{j+1}P_2(T_1 \ge t_j).$$

Let A_{j+1} denote the event that the process hits 1 at some time $t \in [t_j, t_{j+1}]$. We have, for $j \ge j_0 + 1$,

$$\begin{aligned} P_2(T_1 \ge t_j) &\le P_2(\cap_{i=j_0}^{j-1} A_{i+1}^c) \\ &\le \prod_{i=j_0}^{j-1} \Big(1 - P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}(T_1 \le t_{i+1} - t_i)\Big). \end{aligned} \tag{13}$$

If we show that

$$\lim_{j\to\infty} P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}(T_1 \le t_{j+1} - t_j) = 1,$$

then it will certainly follow that

$$E_2T_1 < \infty,$$

thereby proving positive recurrence. In order to prove this, it suffices from

$$\begin{aligned} &P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}(T_1 \le t_{j+1} - t_j) \\ \ge\ & 1 - \Big(P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}(T_{f(t_{j+1})} < T_1)\Big)^{l_j} - P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}(\sigma_{l_j}^{(j)} > t_{j+1} - t_j). \end{aligned}$$

to prove that for some choice of positive integers $\{l_j\}_{l=j_0}^{\infty}$,

$$\lim_{j\to\infty} \Big(P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}(T_{f(t_{j+1})} < T_1)\Big)^{l_j} = 0, \tag{14}$$

and

$$\lim_{j\to\infty} P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}(\sigma_{l_j}^{(j)} > t_{j+1} - t_j) = 0. \tag{15}$$

According to the standard probabilistic potential theory, we have

$$\begin{aligned} &P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}(T_{f(t_{j+1})} < T_1) \\ =\ & \frac{\phi(1) - \phi(f(t_j))}{\phi(1) - \phi(f(t_{j+1}))} \\ =\ & 1 - \frac{\phi(f(t_j)) - \phi(f(t_{j+1}))}{\phi(1) - \phi(f(t_{j+1}))}, \end{aligned} \tag{16}$$

Here, we have $\phi(x)$ as in the proof of Theorem 1, that is,

$$\phi(x) := \int_x^{\infty} t^{-\frac{2a}{\sigma^2}} e^{\frac{2b}{\sigma^2}(t-1)} dt.$$

Thus, for all sufficiently large j, by combining (16) with

$$\frac{\phi(f(t_j)) - \phi(f(t_{j+1}))}{\phi(1) - \phi(f(t_{j+1}))} \ge C_1\phi(f(t_j)) \ge C_2 j^{-\frac{2a}{\sigma^2}} e^{\frac{2b}{\sigma^2}(j-1)},$$

we obtain

$$\begin{aligned}
& P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}(T_{f(t_{j+1})} < T_1) \\
= \;& 1 - \frac{\phi(f(t_j)) - \phi(f(t_{j+1}))}{\phi(1) - \phi(f(t_{j+1}))} \\
\leq \;& 1 - C_2 j^{-\frac{2a}{\sigma^2}} e^{\frac{2b}{\sigma^2}(j-1)},
\end{aligned}$$

Hence,

$$\begin{aligned}
\left(P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}(T_{f(t_{j+1})} < T_1)\right)^{l_j} & \leq \left(1 - C_2 j^{-\frac{2a}{\sigma^2}} e^{\frac{2b}{\sigma^2}(j-1)}\right)^{l_j} \\
& = \left(1 - \frac{C_2}{j^{\frac{2a}{\sigma^2}} e^{-\frac{2b}{\sigma^2}(j-1)}}\right)^{l_j}.
\end{aligned}$$

We choose

$$l_j := \left[j^{\frac{2a}{\sigma^2}} \log j e^{-\frac{2b}{\sigma^2}(j-1)}\right].$$

It follows from the fact that

$$\lim_{y\to\infty}(1 - \frac{1}{y})^{yg(y)} = 0, \quad \text{if } \lim_{y\to\infty} g(y) = \infty,$$

that (14) holds. With this choice of l_j, we have, by (11),

$$\begin{aligned}
& P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}(\sigma_{l_j}^{(j)} > t_{j+1} - t_j) \\
\leq \;& e^{\frac{2ab}{\sigma^2}e^j}(2M)^{l_j} \\
= \;& e^{\frac{2ab}{\sigma^2}e^j} e^{j^{\frac{2a}{\sigma^2}} \log j e^{-\frac{2b}{\sigma^2}(j-1)} \log(2M)}.
\end{aligned} \tag{17}$$

Thus, if

$$a > 0 \text{ and } -\frac{2b}{\sigma^2} < 1, \ (i.e., b > -\frac{\sigma^2}{2}),$$

it follows from (17) that

$$\lim_{j\to\infty} P_{f(t_j)}^{CIR;Ref\leftarrow:f(t_{j+1})}(\sigma_{l_j}^{(j)} > t_{j+1} - t_j) = 0.$$

This completes the proof of the theorem. □

4. Transience of the CIR Model When $b < 0$

Theorem 3. *Consider the CIR model corresponding to the generator*

$$\frac{1}{2}\sigma^2 x \frac{d^2}{dx^2} + (a - bx)\frac{d}{dx}$$

in the time-dependent region $[1, f(t)]$, with reflection at both the fixed endpoint and the time-dependent one. Let $\sigma > 0, b < 0$, and $a > 0$ satisfy that there is an $a_0 < \frac{1}{2}$ and an $a \leq a_0$, where a_0 solves $2a = e^{\frac{2b}{\sigma^2} - \frac{2(1+a)}{\sigma^2}}$. Assume that $f(t) \geq \ln t$, for sufficiently large t. If

$$b < -\sigma^2,$$

then the CIR model is transient.

Proof. Let $j_1 = e^2 + 1$; then, $f(j) = \ln j > 2$ for all $j \geq j_1$. Let B_1 be the event that the CIR process hits 1 sometimes between the first time it hits $f(j)$ and the first time it hits $f(t_{j+1})$:

$$B_j := \{X(t) = 1, \text{ for some } t \in (T_{f(t_j)}, T_{f(t_{j+1})})\}.$$

If we show that

$$\sum_{j=j_1}^{\infty} P_1(B_j) < \infty, \tag{18}$$

then, according to the Borel–Cantelli lemma, it will follow that $P_1(B_j, \text{ i.o.}) = 0$, and consequently the CIR process is transient.

To consider whether or not the event B_j occurs, we first wait until time $T_{f(t_j)}$. Hence, we have $T_{f(j)} \geq j$, since $f(j)$ is not accessible to the process before time j. Since we may have $T_{f(j)} < j + 1$, the point $f(j+1)$ may not be accessible to the process at time $T_{f(j)}$. However, when we wait for one unit of time, then after that, the point $f(j+1)$ certainly will be accessible because of $T_{f(j)} + 1 \geq j + 1$.

Let $M_j < f(j) - 1$. So, the process never got to the level $f(j) - M_j$ in that one unit of time; then, the probability of B_j occurring is no more than $P^{CIR;Ref\leftarrow:f(j+1)}_{f(j)-M_j}(T_1 < T_{f(j+1)})$. By comparison with the process that is reflected at the fixed point $f(j)$, the probability that the process will get to the level $f(j) - M_j$ in that one unit of time is bounded from above by $P^{CIR;Ref\leftarrow:f(j)}_{f(j)}(T_{f(j)-M_j} \leq 1)$. From these considerations above, we have

$$P_1(B_j) \leq P^{CIR;Ref\leftarrow:f(j+1)}_{f(j)-M_j}(T_1 < T_{f(j+1)}) + P^{CIR;Ref\leftarrow:f(j)}_{f(j)}(T_{f(j)-M_j} \leq 1). \tag{19}$$

It follows by standard probabilistic potential theory that

$$P^{CIR;Ref\leftarrow:f(j+1)}_{f(j)-M_j}(T_1 < T_{f(j+1)}) = \frac{\phi(f(j) - M_j) - \phi(f(j+1))}{\phi(1) - \phi(f(j+1))}. \tag{20}$$

We choose $M_j = \frac{1}{2}f(j)$ because of $M_j < f(j) - 1$. Recall that $f(j) \geq \log j$. Then, we have

$$\begin{aligned}\phi(f(j) - M_j) &= \phi(\frac{1}{2}f(j)) = \phi(\frac{1}{2}\log j) \\ &\sim -\frac{\sigma^2}{2b}(\frac{1}{2}\log j)^{-\frac{2a}{\sigma^2}} e^{\frac{2b}{\sigma^2}(\frac{1}{2}\log j)} \\ &= -\frac{\sigma^2}{2b}(\frac{1}{2}\log j)^{-\frac{2a}{\sigma^2}} j^{\frac{b}{\sigma^2}}.\end{aligned}$$

By the assumption that $b < -\sigma^2$, we have

$$-\frac{b}{\sigma^2} > 1.$$

Hence, it follows from (20) that

$$\sum_{j=j_1}^{\infty} P^{CIR;Ref\leftarrow:f(j+1)}_{f(j)-M_j}(T_1 < T_{f(j+1)}) < \infty. \tag{21}$$

We now estimate

$$P^{CIR;Ref\leftarrow:f(j)}_{f(j)}(T_{f(j)-M_j} \leq 1),$$

where $M_j = \frac{1}{2}f(j)$. According to Markov's inequality, we have, for $\lambda > 0$,

$$P^{CIR;Ref\leftarrow:f(j)}_{f(j)}(T_{f(j)-M_j} \leq 1) \leq e^{\lambda} E^{CIR;Ref\leftarrow:f(j)}_{f(j)}[e^{-\lambda T_{f(j)-M_j}}]. \tag{22}$$

By comparison, we have

$$E_{f(j)}^{CIR;Ref\leftarrow:f(j)}[e^{-\lambda T_{f(j)-M_j}}] \leq E_{\beta}^{\frac{a-b\alpha}{\sigma^2\beta};Ref\leftarrow:\beta} \exp\Big(-\frac{\lambda}{\sigma^2\beta}T_\alpha\Big). \tag{23}$$

According to Lemma A5 with $\alpha = f(j) - M_j = \frac{1}{2}f(j)$ and $\beta = f(j)$, we have

$$\begin{aligned}
& P_{f(j)}^{CIR;Ref\leftarrow:f(j)}(T_{f(j)-M_j} \leq 1) \\
\leq\ & e^{\lambda}\frac{(r_1+r_2)e^{-2\frac{a-b\beta}{\sigma^2\beta}(\beta-\alpha)}}{r_1e^{-r_1(\beta-\alpha)}+r_2e^{r_2(\beta-\alpha)}} \\
=\ & e^{\lambda}\frac{(r_1+r_2)e^{-2\frac{a-bf(j)}{\sigma^2 f(j)}M_j}}{r_1e^{-r_1M_j}+r_2e^{r_2M_j}} \\
\leq\ & e^{\lambda}\frac{(r_1+r_2)e^{-2\frac{a-bf(j)}{\sigma^2 f(j)}M_j}}{r_2e^{r_2M_j}} \\
=\ & e^{\lambda}(1+\frac{r_1}{r_2})e^{-r_2M_j-2\frac{a-bf(j)}{\sigma^2 f(j)}M_j}.
\end{aligned}$$

Due to the complexity of $\frac{r_1}{r_2}$, we handle it separately. By substituting the specific expressions of r_1 and r_2 from Lemma A5 into $\frac{r_1}{r_2}$, after some calculations, we obtain

$$\begin{aligned}
\frac{r_1}{r_2} =\ & \frac{\sqrt{\Big(\frac{a-b\beta}{\sigma^2\beta}\Big)^2+\frac{2\lambda}{\sigma^2\beta}}+\frac{a-b\beta}{\sigma^2\beta}}{\sqrt{\Big(\frac{a-b\beta}{\sigma^2\beta}\Big)^2+\frac{2\lambda}{\sigma^2\beta}}-\frac{a-b\beta}{\sigma^2\beta}} \\
=\ & \frac{\sqrt{\Big(\frac{a-bf(j)}{\sigma^2 f(j)}\Big)^2+\frac{2\lambda}{\sigma^2 f(j)}}+\frac{a-bf(j)}{\sigma^2 f(j)}}{\sqrt{\Big(\frac{a-bf(j)}{\sigma^2 f(j)}\Big)^2+\frac{2\lambda}{\sigma^2 f(j)}}-\frac{a-bf(j)}{\sigma^2 f(j)}} \\
=\ & \frac{\Big(\sqrt{\Big(\frac{a-bf(j)}{\sigma^2 f(j)}\Big)^2+\frac{2\lambda}{\sigma^2 f(j)}}+\frac{a-bf(j)}{\sigma^2 f(j)}\Big)^2}{\Big(\frac{a-bf(j)}{\sigma^2 f(j)}\Big)^2+\frac{2\lambda}{\sigma^2 f(j)}-(\frac{a-bf(j)}{\sigma^2 f(j)})^2} \\
=\ & \frac{\sigma^2 f(j)}{2\lambda}\Big(\sqrt{\Big(\frac{a-bf(j)}{\sigma^2 f(j)}\Big)^2+\frac{2\lambda}{\sigma^2 f(j)}}+\frac{a-bf(j)}{\sigma^2 f(j)}\Big)^2 \\
=\ & \frac{\sigma^2}{2\lambda}\Big(\sqrt{\Big(\frac{a}{\sigma^2\sqrt{f(j)}}-\frac{b\sqrt{f(j)}}{\sigma^2}\Big)^2+\frac{2\lambda}{\sigma^2}}+\frac{a}{\sigma^2\sqrt{f(j)}}-\frac{b\sqrt{f(j)}}{\sigma^2}\Big)^2 \\
\leq\ & \frac{\sigma^2}{2\lambda}\Big[4(\frac{a}{\sigma^2\sqrt{f(j)}}-\frac{b\sqrt{f(j)}}{\sigma^2})^2+2(\frac{2\lambda}{\sigma^2})\Big] \\
\leq\ & \frac{\sigma^2}{2\lambda}\Big[8(\frac{a}{\sigma^2\sqrt{f(j)}})^2+8(\frac{b\sqrt{f(j)}}{\sigma^2})^2+2(\frac{2\lambda}{\sigma^2})\Big] \\
=\ & \frac{\sigma^2}{2\lambda}\Big[\frac{8a^2}{\sigma^4 f(j)}+\frac{8b^2 f(j)}{4\sigma^4}+\frac{4\lambda}{\sigma^2}\Big] \\
=\ & \frac{1}{\lambda\sigma^2}\Big[\frac{4a^2}{f(j)}+b^2 f(j)+2\lambda^2\sigma^2\Big].
\end{aligned}$$

Hence, we have

$$
\begin{aligned}
& P_{f(j)}^{CIR;Ref\leftarrow:f(j)}(T_{f(j)-M_j}\leq 1) \\
= \;& e^{\lambda}(1+\frac{r_1}{r_2})\exp\Big(-\Big(\sqrt{\Big(\frac{a}{\sigma^2 f(j)}-\frac{b}{\sigma^2}\Big)^2+\frac{2\lambda}{\sigma^2 f(j)}} \\
& \qquad -\Big(\frac{a}{\sigma^2 f(j)}-\frac{b}{\sigma^2}\Big)+2\frac{a-bf(j)}{\sigma^2 f(j)}\Big)M_j\Big) \\
= \;& e^{\lambda}(1+\frac{r_1}{r_2})\exp\Big(-\Big(\sqrt{\Big(\frac{a}{\sigma^2 f(j)}-\frac{b}{\sigma^2}\Big)^2+\frac{2\lambda}{\sigma^2 f(j)}} \\
& \qquad +\Big(\frac{a}{\sigma^2 f(j)}-\frac{b}{\sigma^2}\Big)\Big)M_j\Big) \\
\sim \;& e^{\lambda}(1+\frac{1}{\lambda\sigma^2}\Big[b^2 f(j)+2\lambda^2\Big])\exp\Big(\frac{2b}{\sigma^2}M_j\Big) \\
= \;& e^{\lambda}(1+\frac{1}{\lambda\sigma^2}\Big[b^2 f(j)+2\lambda^2\Big])\exp\Big(\frac{b}{\sigma^2}f(j)\Big) \\
= \;& e^{\lambda}(1+\frac{1}{\lambda\sigma^2}\Big[b^2\ln j+2\lambda^2\Big])j^{\frac{b}{\sigma^2}}.
\end{aligned}
$$

By the assumption $b<-\sigma^2$, we have

$$-\frac{b}{\sigma^2}>1.$$

Then, we have

$$\sum_{j=j_0}^{\infty} P_{f(j)}^{CIR;Ref\leftarrow:f(j)}(T_{f(j)-M_j}\leq 1)<\infty. \tag{24}$$

Now, (19), (21), and (24) give us (18), and this completes the proof of the theorem. □

Remark 2. *By comparing Theorems 1–3, we clearly find that there is a gap for b, that is,* $-\sigma^2\leq b<-\frac{\sigma^2}{2}$*. We expect that the CIR process is also recurrent in this gap of b. However, we cannot confirm this assertion because the estimates we use here cannot guarantee it.*

Last Passage Time

In the transient case, it is natural to consider the last passage time, which is a random time but not a stopping time. In recent years, last passage time has also played an increasing role in financial modeling, such as in models of default risk, models of insider trading, and the prices of European put and call options.

For the case of diffusion in the form of the CIR model, a differentiable increasing scale function is

$$s(x)=\int_c^x \exp\Big(-2\int_c^u \frac{b(v)}{\sigma^2(v)}dv\Big)du$$

for some choice of $c\in(0,\infty)$. Here in the CIR model, the drift coefficient $b(x)=a-bx$ and the diffusion coefficient $\sigma(x)=\sigma\sqrt{x}$; hence, we obtain the scale function

$$s(x)=C\int_c^x u^{-\frac{2a}{\sigma^2}}e^{\frac{2b}{\sigma^2}u}du, \tag{25}$$

as well as the constants $b<0$ and $C=c^{\frac{2a}{\sigma^2}}e^{-\frac{2b}{\sigma^2}c}$.

Theorem 4. *Let X be a transient CIR process in Theorem 3 such that* $X_t\to+\infty$ *when* $t\to\infty$*, and the last time that X hits y is defined as*

$$\Gamma_y:=\sup\{t:X_t=y\}.$$

Then,

$$P_x(\Gamma_y > t|\mathcal{F}_t) = \frac{s(X_t)}{s(y)} \wedge 1,$$

and the scale function is given in (25).

Proof. The following proof is classic and can be found in many classic textbooks of stochastic process, such as in [20,21], as well as [22]. Observe that $P_x(\Gamma_y > t|\mathcal{F}_t) = P_x(\inf_{s\geq t} X_s < y|\mathcal{F}_t)$; it follows from the Markov property of X that

$$P_x\Big(\inf_{s\geq t} X_s < y|\mathcal{F}_t\Big) = P_{X_t}\Big(\inf_{s\geq 0} X_s < y\Big) = P_{X_t}\Big(\sup_{s\geq 0}(-s(X_s)) > -s(y)\Big).$$

In the following fact, let M be a positive continuous local martingale such that $M_0 = x, M_t \geq 0$, and $\lim_{t\to\infty} M_t = 0$; then,

$$\sup_{t\geq 0} M_t \stackrel{\mathrm{d}}{=} \frac{x}{U},$$

where U is a random variable with a uniform law on $[0,1]$. Hence,

$$P_x(\Gamma_y > t|\mathcal{F}_t) = P_{X_t}\Big(\sup_{s\geq 0}(-s(X_s)) > -s(y)\Big) = \frac{s(X_t)}{s(y)} \wedge 1.$$

This completes the proof of this theorem. □

5. Conclusions

This paper studies the transience/recurrence for CIR process when $b < 0$. By adding boundaries to a time-dependent domain, we obtained a CIR process when $b < 0$ with the transient property that became a CIR with a property of recurrence; however, the boundaries continue to grow over time. We have specified the conditions that the coefficients of the CIR process must meet when it is recurrent, positive recurrent and transient.

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Appendix A

We assume that $u'(x) \leq 0$ and $v'(x) \leq 0$ for all $x \in [1,\beta]$.

Lemma A1. *For $\lambda > 0$,*

$$\begin{cases} \frac{1}{2}u'' + bx^{\gamma}u' + \lambda u = 0, & in\ (1,\beta) \\ u'(1) = 0, \\ u(\beta) = 1, \end{cases} \tag{A1}$$

and

$$\begin{cases} \frac{1}{2}v'' + Dv' + \lambda v = 0, & in\ (1,\beta) \\ v'(1) = 0, \\ v(\beta) = 1, \end{cases} \tag{A2}$$

Then, $D \leq \min_{x\in[1,\beta]} bx^\gamma$ implies that

$$u(x) \leq v(x), \text{ for all } x \in [1, \beta].$$

Proof. Assume that the conclusion is false. According to $u, v \in C^2$, we assume $u(x) > v(x)$ for all $x \in [1, \beta)$ WLOG.

So, $u(1) > v(1), u'(1) = v'(1) = 0$ implies $u''(1) < v''(1)$ according to (A3) and (A4). Then, it follows that $u'(x) < v'(x)$ in $(1, c)$, with $1 < c \leq \beta$ and $u'(c) = v'(c)$. Then, it follows from

$$\frac{u'(c) - u'(c-h)}{h} > \frac{v'(c) - v'(c-h)}{h}$$

that $u''(c) \geq v''(c)$.

$u(c) > v(c), u'(c) = v'(c)$ and $u''(c) \geq v''(c)$, which are contradictory to (A3) and (A4). □

Lemma A2. *For $\lambda > 0$,*

$$\begin{cases} \frac{1}{2}u'' + bx^\gamma u' + \lambda_u u = 0, & in\ (1, \beta) \\ u'(1) = 0, \\ u(\beta) = 1, \end{cases} \tag{A3}$$

and

$$\begin{cases} \frac{1}{2}v'' + Dv' + \lambda_v v = 0, & in\ (1, \beta) \\ v'(1) = 0, \\ v(\beta) = 1, \end{cases} \tag{A4}$$

Then, $D \leq \min_{x\in[1,\beta]} bx^\gamma$ and $\lambda_u \geq \lambda_v$ imply that

$$u(x) \leq v(x), \text{ for all } x \in [1, \beta].$$

Proof. This proof is similar to the one in Lemma A1, so we omit it. □

Lemma A3. *Let u be the solution to the ODE*

$$\begin{cases} \frac{1}{2}u'' + \frac{a-bx}{\sigma^2 x}u' - \frac{\lambda}{\sigma^2 x}u = 0, & in\ (\alpha, \beta) \\ v(\alpha) = 1, \\ v'(\beta) = 0, \end{cases} \tag{A5}$$

and let v be the solution to the ODE

$$\begin{cases} \frac{1}{2}v'' + \frac{a-b\beta}{\sigma^2 \beta}v' - \frac{\lambda}{\sigma^2 \beta}v = 0, & in\ (\alpha, \beta) \\ v(\alpha) = 1, \\ v'(\beta) = 0. \end{cases} \tag{A6}$$

Then,

$$u(x) \leq v(x), \forall x \in [\alpha, \beta].$$

Proof. Assume that the conclusion is false. According to $u, v \in C^2$, we assume $u(x) > v(x)$ for all $x \in (\alpha, \beta]$ WLOG.

So, $u(\beta) > v(\beta), u'(\beta) = v'(\beta) = 0$ implies that $u''(\beta) < v''(\beta)$ according to (A5) and (A6). Then, it follows that $u'(x) < v'(x)$ in (c, β), with $\alpha \leq c < \beta$ and $u'(c) = v'(c)$. Then, it follows from

$$\frac{u'(c) - u'(c+h)}{h} > \frac{v'(c) - v'(c+h)}{h}$$

that $u''(c) \geq v''(c)$.

$u(c) > v(c) > 0$ implies that $-\frac{\lambda}{\sigma^2\beta}v(c) < -\frac{\lambda}{\sigma^2 x}u(x)|_{x=c}$, and $u'(c) = v'(c) < 0$ implies that $\frac{a-b\beta}{\sigma^2\beta}v'(c) \leq \frac{a-bx}{\sigma^2 x}u'(x)|_{x=c}$, together with $u''(c) \geq v''(c)$, which are contradictory to (A5) and (A6). □

Let $T_\alpha = \inf\{t \geq 0 : X(t) = \alpha\}$. Let

$$L^{CIR} = \frac{1}{2}\sigma^2 x\frac{d^2}{dx^2} + (a - bx)\frac{d}{dx},$$

and

$$L^{\Gamma} = \frac{1}{2}\frac{d^2}{dx^2} + \Gamma\frac{d}{dx}.$$

Let $P_x^{CIR;Ref\leftarrow:\beta}$ and $E_x^{CIR;Ref\leftarrow:\beta}$ denote the probabilities and expectations, respectively, for the CIR model corresponding to L^{CIR} on $[1, \beta]$, starting from $x \in [1, \beta]$, with reflection at β and stopped at 1; let $P_x^{CIR;Ref\rightarrow:\alpha}$ and $E_x^{CIR;Ref\rightarrow:\alpha}$ denote the probabilities and expectations, respectively, for the CIR model corresponding to L^{CIR} on $[\alpha, \infty)$, starting from $x \in [\alpha, \infty]$, with reflection at α. We will sometimes work for L^{Γ} with only a constant drift, which we will denote by Γ, in which case Γ will replace the CIR in all of the above notions.

The following lemma comes from the Proposition 2.3 in [10]; for the convenience of readers, we will now provide a proof of this lemma.

Lemma A4. *For $\lambda > 0$ and $1 < \alpha < \beta$,*

$$E_\beta^{\Gamma;Ref\leftarrow:\beta}\exp(-\lambda T_\alpha) = \frac{(r_1 + r_2)e^{-2\Gamma(\beta-\alpha)}}{r_1 e^{-r_1(\beta-\alpha)} + r_2 e^{r_2(\beta-\alpha)}},$$

where $r_1 = \sqrt{\Gamma^2 + 2\lambda} + \Gamma$ and $r_2 = \sqrt{\Gamma^2 + 2\lambda} - \Gamma$.

Proof. According to the Feynman–Kac formula, for any $x \in [\alpha, \beta]$,

$$w(x) = E_x^{\Gamma;Ref\leftarrow:\beta}\exp(-\lambda T_\alpha),$$

solves the boundary value problem $(L^{\Gamma} - \lambda)w = 0$ in (α, β), with the Dirichlet boundary condition at α and the Neumann boundary condition at β, that is,

$$\begin{cases} (L_\Gamma - \lambda)w = 0, & \text{in } (\alpha, \beta) \\ w(\alpha) = 1, \\ w'(\beta) = 0, \end{cases} \tag{A7}$$

The solution of this linear equation is given by

$$w(x) = \frac{r_1 e^{-r_1(\beta-\alpha)}e^{r_2(x-\alpha)} + r_2 e^{r_2(\beta-\alpha)}e^{-r_1(x-\alpha)}}{r_1 e^{-r_1(\beta-\alpha)} + r_2 e^{r_2(\beta-\alpha)}},$$

where $r_1 = \sqrt{\Gamma^2 + 2\lambda} + \Gamma$ and $r_2 = \sqrt{\Gamma^2 + 2\lambda} - \Gamma$.

Substituting $x = \beta$ completes the proof. □

Lemma A5. *For $\lambda > 0$ and $1 < \alpha < \beta$,*

$$E_\beta^{\frac{a-b\beta}{\sigma^2\beta};Ref\leftarrow:\beta}\exp\left(-\frac{\lambda}{\sigma^2\beta}T_\alpha\right) = \frac{(r_1 + r_2)e^{-2\frac{a-b\beta}{\sigma^2\beta}(\beta-\alpha)}}{r_1 e^{-r_1(\beta-\alpha)} + r_2 e^{r_2(\beta-\alpha)}},$$

where

$$r_1 = \sqrt{\left(\frac{a-b\beta}{\sigma^2\beta}\right)^2 + \frac{2\lambda}{\sigma^2\beta}} + \frac{a-b\beta}{\sigma^2\beta}$$

and

$$r_2 = \sqrt{\Big(\frac{a-b\beta}{\sigma^2\beta}\Big)^2 + \frac{2\lambda}{\sigma^2\beta}} - \frac{a-b\beta}{\sigma^2\beta}.$$

Proof. This lemma can be directly obtained from Lemma A4 by simply replacing λ with $\frac{\lambda}{\sigma^2\beta}$ and replacing Γ with $\frac{a-b\beta}{\sigma^2\beta}$. □

Appendix B

Lemma A6. *For $x \in [1,\beta]$ and $\lambda = -\frac{2ab}{\sigma^2} > 0$,*

$$u(x) = e^{-\frac{2b}{\sigma^2}(\beta - x)} \exp\left(- \int_x^\beta \frac{\exp(-\frac{2b(y-1)}{\sigma^2})y^{-\frac{2a}{\sigma^2}}}{\int_1^y \exp(-\frac{2b(t-1)}{\sigma^2})t^{-\frac{2a}{\sigma^2}}dt - \frac{\sigma^2}{2b}} dy \right) \tag{A8}$$

solves the following equation

$$\begin{cases} (L_{CIR} + \lambda)u = 0, & in\ (1,\beta) \\ u(\beta) = 1, \\ u'(1) = 0, \end{cases} \tag{A9}$$

that is,

$$\frac{1}{2}\sigma^2 x u'' + (a - bx)u' + \lambda u = 0$$

with the Dirichlet boundary condition at β and $u(\beta) = 1$ and the Neumann boundary condition at 1 and $u'(1) = 0$.

Proof. For the eigenvalue $\lambda = -\frac{2ab}{\sigma^2} > 0$ (due to $b < 0$), obviously, $u(x) = e^{\frac{2b}{\sigma^2}x}$ is a eigenfunction of Equation (A9).

Using the transformation

$$r(x) = \frac{u'(x)}{u(x)}, \text{ i.e., } u(x) = \exp\Big(\int_1^x r(t)dt\Big),$$

the linear differential equation of the second order

$$\frac{1}{2}\sigma^2 x u'' + (a - bx)u' + \lambda u = 0,$$

i.e.,

$$u'' + \frac{a - bx}{\frac{1}{2}\sigma^2 x}u' + \frac{\lambda}{\frac{1}{2}\sigma^2 x}u = 0$$

can be transformed into the Riccati differential equation

$$r' + r^2 + \frac{a - bx}{\frac{1}{2}\sigma^2 x}r + \frac{\lambda}{\frac{1}{2}\sigma^2 x} = 0,$$

where $\lambda = -\frac{2ab}{\sigma^2} > 0$. Obviously, $r_0 = \frac{2b}{\sigma^2}$ is a solution of the Riccati equation. If a solution r_0 of the Riccati equation is known, then all of the other solutions can be obtained in the form

$$r(x) = r_0 + \frac{1}{z(x)},$$

where $z(x)$ ia an arbitrary solution of the following linear equation

$$z' - \Big[\frac{a - bx}{\frac{1}{2}\sigma^2 x} + 2r_0\Big]z = 1.$$

Since $u'(1) = 0$, we have $r(1) = \frac{u'(1)}{u(1)} = 0$; hence,

$$z(1) = -\frac{1}{r_0} = -\frac{\sigma^2}{2b}.$$

Next, we will solve the following Bernoulli's equation

$$z' - \Big[\frac{2(a-bx)}{\sigma^2 x} + \frac{4b}{\sigma^2}\Big]z = 1,$$

with the Dirichlet boundary condition at 1, i.e., $z(1) = -\frac{\sigma^2}{2b}$. The general solution to the homogeneous equation is

$$z_0(x) = C\exp\Big(\frac{2b(x-1)}{\sigma^2}\Big)x^{\frac{2a}{\sigma^2}},$$

from which a particular solution z_1 of the nonhomogeneous equation can be obtained

$$z_1(x) = \exp(\frac{2b(x-1)}{\sigma^2})x^{\frac{2a}{\sigma^2}}\int_1^x \exp(-\frac{2b(y-1)}{\sigma^2})y^{-\frac{2a}{\sigma^2}}dy.$$

Thus, $z(1) = -\frac{\sigma^2}{2b}$, $C = -\frac{\sigma^2}{2b}$ can be immediately obtained. The general solution of the Bernoulli's equation is

$$z(x) = z_0(x) + z_1(x)$$

with $C = -\frac{\sigma^2}{2b}$.

Hence, the general solution of the original Riccati equation is now obtained in the form

$$\begin{aligned} r(x) &= r_0 + \frac{1}{z(x)} = r_0 + \frac{1}{z_0(x)+z_1(x)} \\ &= \frac{2b}{\sigma^2} + \frac{\exp(-\frac{2b(x-1)}{\sigma^2})x^{-\frac{2a}{\sigma^2}}}{\int_1^x \exp(-\frac{2b(t-1)}{\sigma^2})t^{-\frac{2a}{\sigma^2}}dt - \frac{\sigma^2}{2b}}. \end{aligned} \tag{A10}$$

So, we obtain the solution of the linear differential equation of the second order (A9):

$$\begin{aligned} u(x) &= C\exp\Big(\int_1^x r(y)dy\Big) \\ &= Ce^{\frac{2b}{\sigma^2}(x-1)}\exp\Bigg(\int_1^x \frac{\exp(-\frac{2b(y-1)}{\sigma^2})y^{-\frac{2a}{\sigma^2}}}{\int_1^y \exp(-\frac{2b(t-1)}{\sigma^2})t^{-\frac{2a}{\sigma^2}}dt - \frac{\sigma^2}{2b}}dy\Bigg). \end{aligned}$$

Since $u(\beta) = 1$,

$$C = e^{-\frac{2b}{\sigma^2}(\beta-1)}\exp\Bigg(-\int_1^\beta \frac{\exp(-\frac{2b(y-1)}{\sigma^2})y^{-\frac{2a}{\sigma^2}}}{\int_1^y \exp(-\frac{2b(t-1)}{\sigma^2})t^{-\frac{2a}{\sigma^2}}dt - \frac{\sigma^2}{2b}}dy\Bigg)$$

can be obtained. Therefore, we obtain that

$$u(x) = e^{-\frac{2b}{\sigma^2}(\beta-x)}\exp\Bigg(-\int_x^\beta \frac{\exp(-\frac{2b(y-1)}{\sigma^2})y^{-\frac{2a}{\sigma^2}}}{\int_1^y \exp(-\frac{2b(t-1)}{\sigma^2})t^{-\frac{2a}{\sigma^2}}dt - \frac{\sigma^2}{2b}}dy\Bigg),$$

solves the Equation (A9). This completes the proof of the lemma. □

Lemma A7. *For $x \in [1, \beta]$ and $\lambda_j = -\frac{2ab}{\sigma^2 j} > 0$, then for all $j \geq 1$,*

$$u(x) = e^{-\frac{2b}{\sigma^2 j}(\beta - x)} \exp\left(-\int_x^\beta \frac{\exp(-\frac{2b(y-1)}{\sigma^2 j}) y^{-\frac{2a}{\sigma^2}}}{\int_1^y \exp(-\frac{2b(t-1)}{\sigma^2 j}) t^{-\frac{2a}{\sigma^2}} dt - \frac{\sigma^2 j}{2b}} dy \right) \tag{A11}$$

solves the following equation

$$\begin{cases} (L_{CIR;j} + \lambda_j)u = 0, & in\ (1, \beta) \\ u(\beta) = 1, \\ u'(1) = 0, \end{cases} \tag{A12}$$

that is

$$\frac{1}{2}\sigma^2 x u'' + (a - \frac{b}{j}x)u' + \lambda_j u = 0$$

with the Dirichlet boundary condition at β and $u(\beta) = 1$ and the Neumann boundary condition at 1 and $u'(1) = 0$.

Proof. This lemma can be directly obtained from Lemma A6. □

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