

Article

Probabilistic Interval-Valued Fermatean Hesitant Fuzzy Set and Its Application to Multi-Attribute Decision Making

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Abstract: It is difficult to describe the hesitation and uncertainty of experts by single-valued information, and the differences in the importance of attributes are often ignored during the decision-making process. This paper introduces the probability and interval values into Fermatean hesitant fuzzy set (FHFS) and creatively proposes the probabilistic interval-valued Fermatean hesitant fuzzy set (PIVFHFS) to deal with information loss. This new fuzzy set allows decision makers to use interval-valued information with probability to express their quantitative evaluation, which broadens the range of information expression, effectively reflects the important degree of different membership degrees, and can describe uncertain information more completely and accurately. Under the probabilistic interval-valued Fermatean hesitant fuzzy environment, several new aggregation operators based on Hamacher operation are proposed, including the probabilistic interval-valued Fermatean hesitant fuzzy Hamacher weighted averaging (PIVFHFHWA) operator and geometric (PIVFHFHWG) operator, and their basic properties and particular forms are studied. Then, considering the general correlation between different attributes, this paper defines the probabilistic interval-valued Fermatean hesitant fuzzy Hamacher Choquet integral averaging (PIVFHFHCIA) operator and geometric (PIVFHFHCIG) operator and discusses related properties. Finally, a multi-attribute decision-making (MADM) method is presented and applied to the decision-making problem of reducing carbon emissions of manufacturers in the supply chain. The stability and feasibility of this method are demonstrated by sensitivity analysis and comparative analysis. The proposed new operators can not only consider the correlation between various factors but also express the preference information of decision makers more effectively by using probability, thus avoiding information loss in decision-making progress to some extent.

Keywords: probabilistic interval-valued Fermatean hesitant fuzzy set; Fermatean fuzzy set; Hamacher operation; Choquet integral; multi-attribute decision-making

MSC: 03B52; 03E72; 94D05



Citation: Ruan, C.; Chen, X. Probabilistic Interval-Valued Fermatean Hesitant Fuzzy Set and Its Application to Multi-Attribute Decision Making. *Axioms* **2023**, *12*, 979. <https://doi.org/10.3390/axioms12100979>

Academic Editor: Darjan Karabašević

Received: 5 September 2023

Revised: 5 October 2023

Accepted: 14 October 2023

Published: 17 October 2023



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1. Introduction

In our daily life, there are many fuzzy concepts in human thinking. The characteristics of the objects described by these fuzzy concepts need to be more precise, and the subordination relationship between objects and sets sometimes needs to be clarified. Examples include terms like “Big”, “Warm”, and “Comfortable”. Consequently, the properties of these objects cannot be described simply as “Yes” or “No”. With the rapid development of society and economy, as well as the complexity and uncertainty of evaluation factors, it is difficult for evaluators to give evaluation attitudes for all indicators accurately in reality. Fuzzy sets (FSs) [1] theory can effectively deal with this change, allowing the membership of an object to a set to be in the interval $[0, 1]$ instead of always 0 or 1. Compared with traditional mathematical models, FSs are more effective in solving decision problems

with missing or inaccurate information. In some cases, however, it is difficult to describe the evaluation information and experts' opposition and hesitation only by the degree of membership. Therefore, Atanassov [2] proposed the intuitionistic fuzzy set (IFS) as a generalization of FSs. IFS can use the membership and non-membership functions to express the support and opposition attitude of decision makers to the scheme, respectively, and their complement to 1 is the so-called degree of hesitancy. Thus, IFS can more effectively capture the vagueness of uncertain information and has been widely used in decision-making [3] and expanded into numerous forms, such as intuitionistic fuzzy rough set [4], intuitionistic hesitant fuzzy set [5], linguistic IFS [6], and so on. Then, considering the subjective preference and hesitant attitude of decision makers, Torra [7] introduced the idea of hesitant fuzzy set (HFS), allowing the membership degree set to contain several possible values. HFS can describe the hesitant preferences of evaluators more effectively, thus forming a more reasonable and practical decision-making process. With the increasing ambiguity of problems and information uncertainty, it is challenging for IFS to meet the expanding complicated decision requirements. Then, Yager [8] developed Pythagorean fuzzy set (PFS), which extends the condition of IFS to the sum of squares of the membership and non-membership degrees less than or equal to 1. With the extensive application of PFS in many fields, great achievements have been made in the aggregation operators, information measures, and decision-making models of PFS.

In 2017, Yager [9] proposed the concept of q -rung orthopair fuzzy set (q -ROFS) based on IFS and PFS, where the sum of q^{th} powers of the membership and non-membership degrees is required to be less than or equal to 1, that is, $0 \leq \mu^q + \nu^q \leq 1$. Obviously, the acceptable geometric area increases with the value of q , so q -ROFS can better indicate (which better indicates) the preferences and ideas of decision makers. Compared with IFS and PFS, q -ROFS is more flexible and general in handling uncertain information. Senapati and Yager [10] then set $q = 3$ and investigated the Fermatean fuzzy set (FFS), where the cubic sum of the membership and non-membership degrees of each element is less than or equal to 1. FFS is a new generalization of q -ROFS through definition, requiring the sum of the third power to be defined in $[0, 1]$. It is clear that FFS has larger decision space, more unrestricted expression of evaluation information, and a more robust ability to process information than IFS and PFS, making it increasingly popular across various fields. Although q -ROFS can describe more generalized uncertainties than FFS, it is not conducive to studying specific cases where q takes different values. FFS has attracted much attention since its appearance because of its stronger ability to describe fuzziness. Senapati and Yager [11] defined some basic operations on FFS and introduced four new weighted aggregated operators under the Fermatean fuzzy environment. Aydemir and Gunduz [12] defined several Fermatean fuzzy aggregation operators using the Dombi operation. Liu et al. [13] proposed Fermatean fuzzy linguistic term sets (FFLTs) and developed some weighted aggregation operators and distance measures for FFLTss. Hadi et al. [14] proposed some Fermatean fuzzy Hamacher arithmetic and geometric aggregation operators. Deng and Wang [15] investigated a novel Fermatean fuzzy entropy measure to describe the fuzziness degree of FFS, which considered the uncertainty information and uncertainty degree in FFS. Zeb et al. [16] described the application of new aggregation operators in the Fermatean fuzzy soft sets (FFSSs) environment. Ganie [17] introduced some novel distance measures for FFSs using t -conorms and developed new knowledge measures of FFS with the help of the suggested distance measures. Kirisci [18] presented an extended version of the ELimination Et Choix Traduisant la REalité (ELECTRE) I method under the Fermatean fuzzy environment for solving multi-criteria group decision-making (MCGDM) problems. To solve the problem of multiple membership degrees under Fermatean fuzzy environment, Ruan et al. [19] introduced the concept of Fermatean hesitant fuzzy set (FHFS) and proposed a prioritized Heronian mean operator for FHFSs. Mishra [20] defined several distance measures of FHFSs and developed the remoteness index-based Fermatean hesitant fuzzy-VIKOR MADM method. Wang et al. [21] proposed some hesitant Fermatean fuzzy Bonferroni mean operators for multi-attribute decision-making problems. In conclusion,

FFS has made some achievements in aggregation operators, information measures, decision methods, and so on.

However, in cases where the evaluators need more expertise or the objective decision condition is not ideal, decision makers cannot express their agreement or disagreement with one or more specific numbers. It is more appropriate for decision makers to use interval numbers instead of concrete values to describe evaluation information. Therefore, some scholars began to combine fuzzy set theory with interval numbers to develop new fuzzy sets, including interval-valued IFS (IVIFS), interval-valued PFS (IVPFS), and interval-valued HFS (IVHFS). These new fuzzy sets can express the evaluation information better and deal with the missing information more effectively. Atanassov and Gargov [22] introduced the idea of IVIFS. Then, Atanassov [23] defined different operators of IVIFS and studied their basic properties. Nayagam and Sivaraman [24] introduced a new method for ranking IVIFS. Chen et al. [25] extended the HFS to an interval-valued environment, where the membership of an element to a given set is represented by several possible interval values, and established the interval-valued hesitant preference relation. Peng and Yang [26] defined several aggregation operators under the interval-valued Pythagorean fuzzy environment. Zhang et al. [27] investigated the interval-valued Pythagorean hesitant fuzzy set (IVPHFS), which can preserve the interval-valued fuzzy information as much as possible, and proposed its score function and accuracy function. Since interval-valued Fermatean fuzzy sets (IVFFSs) are more flexible and reliable tools for dealing with uncertain and incomplete information, scholars have extended interval numbers to the Fermatean fuzzy environment and achieved some results in recent years. Jeevaraj [28] introduced the idea of IVFFSs as an extension of IFS and developed various score functions in the class of IVFFSs. Akram [29] demonstrated an interval-valued Fermatean fuzzy fractional transportation problem. Mishra et al. [30] introduced interval-valued FHFSS (IVFHFSS) and discussed a decision analysis process on IVFHFSS environment based on the COPRAS method. Qin et al. [31] proposed a novel score function for IVFFSs and constructed a new multi-attribute decision-making (MADM) method using the hybrid weighted score measure. Sergi et al. [32] proposed a new fuzzy extension of the most-used capital budgeting techniques with IVFFSs information. Rani and Mishra [33] developed the doctrine of IVFFSs and their fundamental operations. Demir [34] developed four different types of correlation coefficients for FHFSS and extended them to the correlation coefficients and weighted correlation coefficients for IVFHFSSs.

Individual uncertainty and limited knowledge can negatively impact decision-making processes and ultimately affect the rationality of results due to potential information loss. When evaluation information is insufficient, IVHFS is limited in accurately describing the probability of evaluating information, so some scholars introduced the probability into different FSs. Zhang et al. [35] proposed the probabilistic hesitant fuzzy set (PHFS), which can retain more information than HFS. Jiang and Ma [36] introduced some new basic operations on probabilistic hesitant fuzzy elements (PHFEs) and developed probabilistic hesitant fuzzy weighted arithmetic and geometric aggregation operators. Li and Wang [37] defined the probabilistic hesitant fuzzy preference relation (PHFPR) based on expected multiplicative consistency transitivity. Later, scholars continued to extend FSs theory by combining probability and interval values. De et al. [38] developed an interactive method for solving decision-making problems with incomplete weight information using probabilistic interval-valued intuitionistic hesitant fuzzy set (PIVIHFS). Garg [39] proposed some probabilistic aggregation operators with Pythagorean fuzzy information and extended them to the IVPFS environment to develop corresponding operators. Ali et al. [40] constructed a probabilistic interval-valued hesitant fuzzy set (PIVHFS)-Technique for Order Preference by Similarity to an Ideal Solution (TOPSIS) model and improved some preliminary aggregation operators based on PIVHFS.

As an important part of FSs theory, aggregation operator is regarded as a valuable tool for solving fuzzy decision-making problems, and it has yielded fruitful results in various fuzzy environments. Many existing operators are improved on the operation of Archimedean t-norm and t-conorm, such as Bonferroni mean (BM) operator [41], Ein-

stein aggregation operators [42], ordered weighted averaging (OWA) operator [43], and Hamacher operators [44]. Among them, Hamacher operator, as a generalization of algebraic and Einstein t-norms and t-conorms, has been widely used in various FSs. Since it can choose different parameters according to the personal preferences of evaluators to deal with decision problems, it is more universal and practical in real life. Many scholars have studied the applications of Hamacher operator in different interval-valued fuzzy environments. Li and Peng [44] proposed some new Hamacher operations for IVHFS. Liu [45] developed several weighted averaging Hamacher operators for IIVFS. Xiao [46] defined an induced interval-valued intuitionistic fuzzy Hamacher ordered weighted geometric (I-IVIFHOWG) operator and applied it to evaluate the security of a wireless sensor network. Senapati and Chen [47] formulated several aggregation operators based on Hamacher triangular norms of IVPFS. Shahzadi [48] introduced several Hamacher initiative weighted averaging aggregation operators for IVFFSs.

With the development of research, scholars found a general correlation between different attributes. Attributes are often not independent of each other but complement or duplicate. Murofushi and Sugeno [49] introduced the fuzzy measure to simulate the interaction between different decision-making criteria. The Choquet integral operator proposed by Choquet [50] fully considered the importance and interaction between attributes and provided an effective solution to this issue. Since its appearance, the Choquet integral operator has been extended to different fuzzy environments. Zhang and Yu [51] defined some geometric Choquet aggregation operators using Einstein operations under the intuitionistic fuzzy environment. Khan et al. [52] proposed several Choquet integral averaging and geometric operators with Pythagorean hesitant fuzzy information. Qu et al. [53] developed some Choquet ordered operators for interval-valued dual hesitant fuzzy sets (IVDHFSs). Khan et al. [54] investigated an interval-valued Pythagorean fuzzy Choquet integral geometric (IVPFCIG) operator based on fuzzy measures for solving MCGDM problems. Luo and Liu [55] used the Hamacher operation and Choquet interval to develop the probabilistic interval-valued hesitant Pythagorean fuzzy Hamacher Choquet integral geometric (PIVHPFHCIG) operator. Shao et al. [56] proposed the probabilistic neutrosophic hesitant fuzzy Choquet averaging and geometric operators to select the third-party logistics providers.

Due to the limitation of their knowledge and experience, it is difficult for experts to evaluate complex decision-making problems accurately, and they often use fuzzy numbers to express their preferences. To make evaluating information more comprehensive and effective, scholars continued expanding their research on FSs and proposed IFS, HFS, PFS, FFS, FHFS, and so on. Significantly, FHFS broadens the restriction of membership and non-membership degrees and effectively considers the hesitant state of experts in decision-making. However, when evaluators need more expertise or objective decision conditions are not ideal, decision makers cannot express their agreement or disagreement with single values. Interval-valued information is more conducive to reflecting the uncertain application of information, and its fusion with FSs further enhances the ability to deal with uncertain information. Compared with traditional single-valued fuzzy sets mentioned above, interval-valued fuzzy sets can preserve more evaluation information and avoid information loss. To solve the problem of multiple membership degrees with different importance degrees in practice, scholars introduced probabilistic information into various fuzzy environments and proposed many new FSs, such as PIFS, PIVIFS, and PIVHFS.

Although the research of FFS has made some achievements, the research content mainly focuses on the information aggregation method, and it is necessary to explore new extension forms. From the existing research, few scholars have researched information aggregation under the probabilistic interval-valued Fermatean hesitant fuzzy environment. Compared with traditional single-valued fuzzy sets, probabilistic interval-valued Fermatean hesitant fuzzy sets (PIVFHFSs) proposed in this paper can broadly express information and retain more evaluation information. Compared with ordinary interval-valued fuzzy sets, PIFHFSs add the corresponding probability information to each membership degree and overcome the shortcoming that different membership degree has different

importance. Thus, they can reflect the difference in evaluation information more accurately. Therefore, it is necessary to explore the PIVFHFSSs and relevant information aggregation methods, which can further enrich the theoretical research of FFS.

As an indispensable part of decision-making, information aggregation has received attention from numerous scholars. Despite the extensive study of information aggregation and the proposal of several effective aggregation methods, information loss and incomplete information remain unavoidable in the decision-making process. Additionally, aggregation operators are widely used in many fuzzy environments as fundamental elements of information aggregation. However, many aggregation operators and their extended forms assume that attributes are independent and fail to analyze the interaction between attributes. At present, some achievements have been made in the study of aggregation operators of FFS. Still, most of them are based on algebraic sums and products, leading to rational constraints on decision makers. Although many scholars have devised multiple information aggregation approaches for FFSs, the combination of probability and interval has not been employed to describe uncertain information and handle fuzzy issues under a Fermatean hesitant fuzzy setting. Therefore, the contributions of this paper are given as:

- (1) This paper adds corresponding probability information for each membership degree and innovatively proposes the concept of PIVFHFSSs.
- (2) This paper defines two new operators for PIVFHFSSs combined with the Hamacher operation, including the probabilistic interval-valued Fermatean hesitant fuzzy Hamacher weighted averaging (PIVFHFHWA) operator and geometric (PIVFHFHWG) operator.
- (3) Considering the correlation between different attributes, this paper further proposes the probabilistic interval-valued Fermatean hesitant fuzzy Hamacher Choquet integral averaging (PIVFHFHCIA) operator and geometric (PIVFHFHCIG) operator based on Choquet integral.
- (4) Based on the PIVFHFHCIG operator, a MADM model is constructed to solve the carbon emission reduction decision-making problem of manufacturers in the supply chain.

The paper is organized as follows. Section 2 provides a brief review of basic concepts and theories. PIVFHFSSs and related Hamacher operations are defined in Section 3. In Section 4, two Hamacher weighted aggregation operators of PIVFHFSSs are developed. Combining with Choquet integral, the PIVFHFHCIG operator is developed in Section 5. Then, Section 6 presents a MADM method utilizing the PIVFHFHCIG operator and applies it to carbon emission reduction of manufacturers. Section 7 concludes the paper with some remarks.

2. Preliminaries

2.1. Fermatean Fuzzy Sets

This section provides a concise overview of FFS, FHFS, IVFFS, IVFHFS, as well as theories surrounding Hamacher operation and Choquet integral.

Definition 1 [10]. Let X be a non-empty set. A Fermatean fuzzy set (FFS) on X is an object of the following structure:

$$F = \{ \langle x, \mu_F(x), \nu_F(x) \rangle : x \in X \} \quad (1)$$

where $\mu_F(x) : x \rightarrow [0, 1]$ and $\nu_F(x) : x \rightarrow [0, 1]$ are the membership degree and non-membership degree of x in F with $0 < (\mu_F(x))^3 + (\nu_F(x))^3 \leq 1$. For each element x to F , $\pi_F(x) = \sqrt[3]{1 - (\mu_F(x))^3 - (\nu_F(x))^3}$ is the degree of indeterminacy.

Definition 2 [19]. Let X be a non-empty set, then a Fermatean hesitant fuzzy set (FHFS) H defined on X is given as:

$$H = \{ \langle x, \mu_H(x_i), \nu_H(x_i) \rangle : x \in X \} \quad (2)$$

where $\mu_H(x_i) \rightarrow [0, 1]$ and $\nu_H(x_i) \rightarrow [0, 1]$ satisfy the condition $0 < (\mu_H(x_i))^3 + (\nu_H(x_i))^3 \leq 1$, representing the possible membership and non-membership degrees of an element x_i to H , respectively.

Definition 3 [28]. Let $Q[0, 1]$ be the set of all closed sub-intervals of the interval $[0, 1]$. An interval-valued Fermatean fuzzy set (IVFFS) is defined as follows:

$$\tilde{P} = \{ \langle x, \mu_{\tilde{P}}(x), \nu_{\tilde{P}}(x) \rangle : x \in X \} \quad (3)$$

where $\mu_{\tilde{P}}(x) \rightarrow Q[0, 1]$ and $\nu_{\tilde{P}}(x) \rightarrow Q[0, 1]$ denote, respectively, the interval-valued membership degree and interval-valued non-membership degree of x with the condition $0 < \sup_x (\mu_{\tilde{P}}(x))^3 + \sup_x (\nu_{\tilde{P}}(x))^3 \leq 1$.

$\forall x \in X$, $\mu_{\tilde{P}}(x)$, and $\nu_{\tilde{P}}(x)$ are closed intervals, and their lower and upper bounds are, respectively, represented as $\mu_{\tilde{P}}^L(x), \mu_{\tilde{P}}^U(x)$ and $\nu_{\tilde{P}}^L(x), \nu_{\tilde{P}}^U(x)$. Thus, $\tilde{P}$ is an object of another form as follows:

$$\tilde{P} = \left\{ \left\langle x, [\mu_{\tilde{P}}^L(x), \mu_{\tilde{P}}^U(x)], [\nu_{\tilde{P}}^L(x), \nu_{\tilde{P}}^U(x)] \right\rangle : x \in X \right\} \quad (4)$$

For each element $x \in X$, the hesitancy degree of $\pi_{\tilde{P}}(x)$ on $\tilde{P}$ is $\pi_{\tilde{P}}(x) = [\pi_{\tilde{P}}^L(x), \pi_{\tilde{P}}^U(x)] = [\sqrt[3]{1 - (\mu_{\tilde{P}}^L(x))^3 - (\nu_{\tilde{P}}^L(x))^3}, \sqrt[3]{1 - (\mu_{\tilde{P}}^U(x))^3 - (\nu_{\tilde{P}}^U(x))^3}]$. For convenience, this paper denotes an interval-valued Fermatean fuzzy number (IVFFN) as $\tilde{P} = ([\mu_{\tilde{P}}^L(x), \mu_{\tilde{P}}^U(x)], [\nu_{\tilde{P}}^L(x), \nu_{\tilde{P}}^U(x)])$ with $0 < (\mu_{\tilde{P}}^L(x))^3 + (\nu_{\tilde{P}}^L(x))^3 \leq 1$ and $0 < (\mu_{\tilde{P}}^U(x))^3 + (\nu_{\tilde{P}}^U(x))^3 \leq 1$.

Definition 4 [34]. Let $Q[0, 1]$ be the set of all closed sub-intervals of the interval $[0, 1]$. Then, an interval-valued Fermatean hesitant fuzzy set (IVFHFS) $\ddot{E}$ is defined by

$$\ddot{E} = \{ \langle x, h_{\ddot{E}}(x) \rangle : x \in X \} \quad (5)$$

where $h_{\ddot{E}}(x) = \{ \langle x, \mu_{\ddot{E}}(x), \nu_{\ddot{E}}(x) \rangle : x \in X \}$ is an interval-valued Fermatean hesitant fuzzy number (IVFHFN) denoting some possible interval membership and non-membership values of an element x to $\ddot{E}$.

For each element $x \in X$, $\mu_{\ddot{E}}(x) \rightarrow Q[0, 1]$ and $\nu_{\ddot{E}}(x) \rightarrow [0, 1]$ represent the possible membership intervals non-membership intervals of $\ddot{E}$, respectively, and their lower and upper bounds are denoted as $\mu_{\ddot{E}}^L(x), \mu_{\ddot{E}}^U(x)$ and $\nu_{\ddot{E}}^L(x), \nu_{\ddot{E}}^U(x)$ with $0 < (\mu_{\ddot{E}}^L(x))^3 + (\nu_{\ddot{E}}^L(x))^3 \leq 1$ and $0 < (\mu_{\ddot{E}}^U(x))^3 + (\nu_{\ddot{E}}^U(x))^3 \leq 1$.

2.2. Hamacher t-Norm and t-Conorm

Hamacher operator is a wider range of t-norm and t-conorm, which can make the operation more flexible through its own parameter.

Definition 5 [54]. Hamacher t-norm ($\otimes$) and t-conorm ($\oplus$) have the following definitions:

$$T(x, y) = x \otimes y = \frac{xy}{\delta + (1 - \delta)(x + y - xy)} \quad (6)$$

$$\bar{T}(x, y) = x \oplus y = \frac{x + y - xy - (1 - \delta)xy}{1 - (1 - \delta)xy} \quad (7)$$

where $\delta > 0$. Especially, when $\delta = 1$, the Hamacher t-norm and t-conorm are simplified to algebraic t-norm and t-conorm, respectively. That is,

$$T(x, y) = x \otimes y = xy \quad (8)$$

$$\bar{T}(x, y) = x \oplus y = x + y - xy \quad (9)$$

When $\delta = 2$, the Hamacher t -norm and t -conorm are simplified to Einstein t -norm and t -conorm, respectively.

$$T(x, y) = x \otimes y = \frac{xy}{1 + (1 - x)(1 - y)} \quad (10)$$

$$\bar{T}(x, y) = x \oplus y = \frac{x + y}{1 + xy} \quad (11)$$

2.3. Choquet Integral and Fuzzy Measure

Fuzzy measure can represent not only the weight of attribute and attribute set, but also the relation between different attributes in solving MADM problems.

Definition 6 [49]. Let X be a finite set, then fuzzy measure of X is a function $\kappa : P(X) \rightarrow [0, 1]$, which satisfies the conditions as follows:

- (1) $\kappa(X) = 1, \kappa(\emptyset) = 0$;
- (2) $\forall \alpha, \beta \subseteq P(X)$, if $\alpha \subseteq \beta$, then $\kappa(\alpha) \leq \kappa(\beta)$, where $P(X)$ is a power set of X ;
- (3) $\kappa(\alpha \cap \beta) = \kappa(\alpha) + \kappa(\beta) + \gamma \kappa(\alpha) \kappa(\beta)$ for all $\alpha, \beta \in X$ and $\alpha \cap \beta = \emptyset$, where γ denotes the interaction of indicators with $\gamma > -1$.

Let Λ be a finite set with $\bigcup_{i=1}^n \Lambda_i = \Lambda$, then the fuzzy measure of κ under γ is defined as:

$$\kappa(\Lambda) = \begin{cases} \frac{1}{\gamma} \left(\prod_{i=1}^n (1 + \gamma \kappa(\Lambda_i)) - 1 \right), & \gamma \neq 0 \\ \sum_{i=1}^n \kappa(\Lambda_i), & \gamma = 0 \end{cases} \quad (12)$$

where $\Lambda_i \cap \Lambda_j = \emptyset$ ($i \neq j$) for the sub-set with only one element Λ_i . $\kappa(\Lambda_i)$ is called the fuzzy measure and $\kappa_i = \kappa(\Lambda_i)$. When $\kappa(\Lambda) = 1$, then

$$\gamma = \prod_{i=1}^n (1 + \gamma \kappa(\Lambda_i)) - 1 \quad (13)$$

Definition 7 [50]. Let κ be a fuzzy measure and f be a non-negative real number function defined on a finite set X , then the discrete Choquet integral for f with respect to κ is given as:

$$\Lambda_\kappa(f(\Lambda_{(1)}), f(\Lambda_{(2)}), \dots, f(\Lambda_{(n)})) = \sum_{i=1}^n f(\Lambda_{(i)}) [\kappa(\rho_{(i)}) - \kappa(\rho_{(i+1)})] \quad (14)$$

where (i) is a permutation of $f(\Lambda_{(i)})$, which satisfies the condition $f(\Lambda_{(1)}) \leq f(\Lambda_{(2)}) \leq \dots \leq f(\Lambda_{(n)})$, $\rho_{(i)} = \{\Lambda_{(i)}, \Lambda_{(i+1)}, \dots, \Lambda_{(n)}\}$, and $\rho_{(n+1)} = \emptyset$.

3. Probabilistic Interval-Valued Fermatean Hesitant Fuzzy Set

This section creatively develops the concept of PIVFHFSSs and introduces the scoring function and accuracy function and related comparison rules of PIVFHFSSs. In addition, this section presents some Hamacher operations of PIVFHFSSs, providing a significant theoretical basis for the following aggregation operators based on Hamacher t -norm and t -conorm.

3.1. The Probabilistic Interval-Valued Fermatean Hesitant Fuzzy Set

Definition 8. Let X be a finite set. Then, the probabilistic interval-valued Fermatean hesitant fuzzy set (PIVFHFS) on X is given as:

$$A = \left\{ \left(x, \left[\left(\left[\mu_A^L(x), \mu_A^U(x) \right], \left[\nu_A^L(x), \nu_A^U(x) \right] \right), p \right] \right) : x \in X \right\} \quad (15)$$

where PIVFHS comprises a group of probabilistic interval-valued Fermatean hesitant fuzzy elements (PIVFHFEs) represented by $\left[\left(\left[\mu_A^L(x), \mu_A^U(x) \right], \left[\nu_A^L(x), \nu_A^U(x) \right] \right), p \right]$. Every PIVFHFE is a set of pairs composed by a collection of IVFFNs and the probability p , in which the probability is used to indicate the possible degree of its corresponding PIVFHS. When $\left[\left(\left[\mu_A^L(x), \mu_A^U(x) \right], \left[\nu_A^L(x), \nu_A^U(x) \right] \right), p \right]$ is an infinite PIVFHFE, its probabilities should be a continuous probability distribution with $0 \leq p(x) \leq 1$ and $\int_{p \in \text{PIVFHFE}} p(x) dx = 1$. When $\left[\left(\left[\mu_A^L(x), \mu_A^U(x) \right], \left[\nu_A^L(x), \nu_A^U(x) \right] \right), p \right]$ is a finite set, it is

represented as $\left[\left(\left[\mu_{A_{i_l}}^L(x), \mu_{A_{i_l}}^U(x) \right], \left[\nu_{A_{i_l}}^L(x), \nu_{A_{i_l}}^U(x) \right] \right), p_{i_l} \right]$, where $i = 1, 2, \dots, L(\text{PIVFHFE})$, $L(\text{PIVFHFE})$ is a positive integer that describes the quantity of elements contained in PIVFHFE, $0 \leq p_{i_l} \leq 1$ and $\sum_{i=1}^{L(\text{PIVFHFE})} p_{i_l} = 1$. The indeterminacy degree of interval $[\pi_{A_{i_l}}^L(x), \pi_{A_{i_l}}^U(x)]$ is

$$\pi_{A_{i_l}}^L(x) = \sqrt[3]{1 - \left(\mu_{A_{i_l}}^L(x) \right)^3 - \left(\nu_{A_{i_l}}^L(x) \right)^3} \text{ and } \pi_{A_{i_l}}^U(x) = \sqrt[3]{1 - \left(\mu_{A_{i_l}}^U(x) \right)^3 - \left(\nu_{A_{i_l}}^U(x) \right)^3}.$$

Example 1. If a city aims to select a location for constructing a waste incinerator to achieve the goal of becoming a waste-free city, it must conduct thorough field visits to various regions. Following these visits, the government has preliminarily chosen Region Z and plans to conduct further evaluations of the relevant supporting facilities in the area. Let 10 points be full marks; here, we consider the government is 70% sure about related supporting facilities, which could be from 7 to 8.5 but not less than 5 to 6. Therefore, after data normalization, the opinion can be represented as $([0.7, 0.85], [0.5, 0.6], 0.7)$. Here $[\mu_A^L(x), \mu_A^U(x)] = [0.7, 0.85]$ and $[\nu_A^L(x), \nu_A^U(x)] = [0.5, 0.6]$ where $[\pi_A^L(x), \pi_A^U(x)] = [0.554, 0.810]$ and the probability of surety of the supporting facilities is $p_A = 0.7$.

Let $L_1(\text{PIVFHFE})$ and $L_2(\text{PIVFHFE})$ be the amounts of elements in PIVFHFE1 and PIVFHFE2, respectively. For convenience, let $L_1(\text{PIVFHFE}) = L_2(\text{PIVFHFE})$. To compare the sizes of different PIVFHFEs, the following scoring function and accuracy function are defined.

Definition 9. Let X be a finite set and $M_i = \left[\left(\left[\mu_{M_{i_l}}^L(x), \mu_{M_{i_l}}^U(x) \right], \left[\nu_{M_{i_l}}^L(x), \nu_{M_{i_l}}^U(x) \right] \right), p_{i_l} \right]$ be a collection of PIVFHFEs, where $i = 1, 2, \dots, n$, $l = 1, 2, \dots, L(\text{PIVFHFE})$. The scoring function $S(M_i)$ and accuracy function $E(M_i)$ can be computed as:

$$S(M_i) = \sum_{l=1}^{L(\text{PIVFHFE})} p_{i_l} \times \frac{\left(\left(\mu_{M_{i_l}}^L(x) \right)^3 - \left(\nu_{M_{i_l}}^U(x) \right)^3 \right) + \left(\left(\mu_{M_{i_l}}^U(x) \right)^3 - \left(\nu_{M_{i_l}}^L(x) \right)^3 \right)}{2} \quad (16)$$

$$E(M_i) = \sum_{l=1}^{L(\text{PIVFHFE})} p_{i_l} \times \frac{2 - \left(\pi_{M_{i_l}}^L(x) \right)^3 - \left(\pi_{M_{i_l}}^U(x) \right)^3}{2} \quad (17)$$

Definition 10. Let X be a finite set. For any two PIVFHFEs $M_1 = \left[\left(\left[\mu_{M_{1_l}}^L(x), \mu_{M_{1_l}}^U(x) \right], \left[\nu_{M_{1_l}}^L(x), \nu_{M_{1_l}}^U(x) \right] \right), p_{1_l} \right]$ and $M_2 = \left[\left(\left[\mu_{M_{2_k}}^L(x), \mu_{M_{2_k}}^U(x) \right], \left[\nu_{M_{2_k}}^L(x), \nu_{M_{2_k}}^U(x) \right] \right), p_{2_k} \right]$ on X , where $i = 1, 2, \dots, n$, $l = 1, 2, \dots, L(\text{PIVFHFE}_1)$, $k = 1, 2, \dots, L(\text{PIVFHFE}_2)$, then rules for comparison between them are as follows:

- (1) If $S(M_1) > S(M_2)$, then $M_1 \succ M_2$;
- (2) If $S(M_1) < S(M_2)$, then $M_1 \prec M_2$;
- (3) If $S(M_1) = S(M_2)$, then

- (i) If $E(M_1) > E(M_2)$, then $M_1 \succ M_2$;
- (ii) If $E(M_1) = E(M_2)$, then $M_1 \sim M_2$;
- (iii) If $E(M_1) < E(M_2)$, then $M_1 \prec M_2$.

3.2. Hamacher Operations on PIVFHFES

This section establishes several basic Hamacher operations of PIVFHFES.

Definition 11. Let $M_1 = \left[\left(\left[\mu_{M_1}^L(x), \mu_{M_1}^U(x) \right], \left[\nu_{M_1}^L(x), \nu_{M_1}^U(x) \right] \right), p_{1_l} \right]$ and $M_2 = \left[\left(\left[\mu_{M_2}^L(x), \mu_{M_2}^U(x) \right], \left[\nu_{M_2}^L(x), \nu_{M_2}^U(x) \right] \right), p_{2_k} \right]$ be two PIVFHFES on a finite set X , where $l = 1, 2, \dots, L(\text{PIVFHFE}_1)$, $k = 1, 2, \dots, L(\text{PIVFHFE}_2)$, $\delta \in (0, +\infty)$ and $\lambda > 0$. Then, some Hamacher operations between different PIVFHFES are defined as follows:

$$\begin{aligned}
 M_1 \otimes M_2 &= \left\langle \left[\frac{\mu_{M_1}^L \mu_{M_2}^L}{\sqrt[3]{\delta + (1-\delta) \left((\mu_{M_1}^L)^3 + (\mu_{M_2}^L)^3 - (\mu_{M_1}^L)^3 (\mu_{M_2}^L)^3 \right)}}, \frac{\mu_{M_1}^U \mu_{M_2}^U}{\sqrt[3]{\delta + (1-\delta) \left((\mu_{M_1}^U)^3 + (\mu_{M_2}^U)^3 - (\mu_{M_1}^U)^3 (\mu_{M_2}^U)^3 \right)}} \right], \left[\frac{\nu_{M_1}^L \nu_{M_2}^L}{\sqrt[3]{\delta + (1-\delta) \left((\nu_{M_1}^L)^3 + (\nu_{M_2}^L)^3 - (\nu_{M_1}^L)^3 (\nu_{M_2}^L)^3 \right)}}, \frac{\nu_{M_1}^U \nu_{M_2}^U}{\sqrt[3]{\delta + (1-\delta) \left((\nu_{M_1}^U)^3 + (\nu_{M_2}^U)^3 - (\nu_{M_1}^U)^3 (\nu_{M_2}^U)^3 \right)}} \right], p_{1_l} + p_{2_k} \right\rangle \\
 M_1 \oplus M_2 &= \left\langle \left[\frac{(\mu_{M_1}^L)^3 + (\mu_{M_2}^L)^3 - (\mu_{M_1}^L)^3 (\mu_{M_2}^L)^3 - (1-\delta) (\mu_{M_1}^L)^3 (\mu_{M_2}^L)^3}{\delta + (1-\delta) \left(1 - (\mu_{M_1}^L)^3 (\mu_{M_2}^L)^3 \right)}, \frac{(\mu_{M_1}^U)^3 + (\mu_{M_2}^U)^3 - (\mu_{M_1}^U)^3 (\mu_{M_2}^U)^3 - (1-\delta) (\mu_{M_1}^U)^3 (\mu_{M_2}^U)^3}{\delta + (1-\delta) \left(1 - (\mu_{M_1}^U)^3 (\mu_{M_2}^U)^3 \right)} \right], \left[\frac{\nu_{M_1}^L \nu_{M_2}^L}{\sqrt[3]{\delta + (1-\delta) \left((\nu_{M_1}^L)^3 + (\nu_{M_2}^L)^3 - (\nu_{M_1}^L)^3 (\nu_{M_2}^L)^3 \right)}}, \frac{\nu_{M_1}^U \nu_{M_2}^U}{\sqrt[3]{\delta + (1-\delta) \left((\nu_{M_1}^U)^3 + (\nu_{M_2}^U)^3 - (\nu_{M_1}^U)^3 (\nu_{M_2}^U)^3 \right)}} \right], p_{1_l} + p_{2_k} \right\rangle \\
 \lambda M &= \left\langle \left[\frac{(1 + (\delta - 1) (\mu_{M_1}^L)^3)^\lambda - (1 - (\mu_{M_1}^L)^3)^\lambda}{(1 + (\delta - 1) (\mu_{M_1}^L)^3)^\lambda + (\delta - 1) (1 - (\mu_{M_1}^L)^3)^\lambda}, \frac{(1 + (\delta - 1) (\mu_{M_1}^U)^3)^\lambda - (1 - (\mu_{M_1}^U)^3)^\lambda}{(1 + (\delta - 1) (\mu_{M_1}^U)^3)^\lambda + (\delta - 1) (1 - (\mu_{M_1}^U)^3)^\lambda} \right], \left[\frac{\sqrt[3]{\delta} (\mu_{M_1}^L)^\lambda}{\sqrt[3]{(1 + (\delta - 1) (1 - (\mu_{M_1}^L)^3)^\lambda) + (\delta - 1) (1 - (\mu_{M_1}^L)^3)^\lambda}}, \frac{\sqrt[3]{\delta} (\mu_{M_1}^U)^\lambda}{\sqrt[3]{(1 + (\delta - 1) (1 - (\mu_{M_1}^U)^3)^\lambda) + (\delta - 1) (1 - (\mu_{M_1}^U)^3)^\lambda}} \right], p \right\rangle \\
 M^\lambda &= \left\langle \left[\frac{\sqrt[3]{\delta} (\mu_{M_1}^L)^\lambda}{\sqrt[3]{(1 + (\delta - 1) (1 - (\mu_{M_1}^L)^3)^\lambda) + (\delta - 1) (\mu_{M_1}^L)^{3\lambda}}}, \frac{\sqrt[3]{\delta} (\mu_{M_1}^U)^\lambda}{\sqrt[3]{(1 + (\delta - 1) (1 - (\mu_{M_1}^U)^3)^\lambda) + (\delta - 1) (\mu_{M_1}^U)^{3\lambda}}} \right], \left[\frac{(1 + (\delta - 1) (\nu_{M_1}^L)^3)^\lambda - (1 - (\nu_{M_1}^L)^3)^\lambda}{(1 + (\delta - 1) (\nu_{M_1}^L)^3)^\lambda + (\delta - 1) (1 - (\nu_{M_1}^L)^3)^\lambda}, \frac{(1 + (\delta - 1) (\nu_{M_1}^U)^3)^\lambda - (1 - (\nu_{M_1}^U)^3)^\lambda}{(1 + (\delta - 1) (\nu_{M_1}^U)^3)^\lambda + (\delta - 1) (1 - (\nu_{M_1}^U)^3)^\lambda} \right], p \right\rangle
 \end{aligned}$$

where $p_{1_l} + p_{2_k} = (p_{1_l} + p_{2_k}) / (\sum_{l=1}^{L(\text{PIVFHFE}_1)} p_{1_l} + \sum_{k=1}^{L(\text{PIVFHFE}_2)} p_{2_k})$, $l = 1, 2, \dots, L(\text{PIVFHFE}_1)$, $k = 1, 2, \dots, L(\text{PIVFHFE}_2)$.

Obviously, when $\delta = 1$, the probabilistic interval-valued Fermatean hesitant fuzzy Hamacher operation is reduced to an algebraic operation of PIVFHFES. When $\delta = 2$, the probabilistic interval-valued Fermatean hesitant fuzzy Hamacher operation degenerates into an Einstein operation.

Theorem 1. Let M , M_1 , and M_2 be three PIVFHFES, then

- (1) $M_1 \otimes M_2 = M_2 \otimes M_1$;
- (2) $M_1 \oplus M_2 = M_2 \oplus M_1$;
- (3) $\lambda(M_1 \oplus M_2) = \lambda M_1 \oplus \lambda M_2, \lambda > 0$;
- (4) $(M_1 \otimes M_2)^\lambda = M_2^\lambda \otimes M_1^\lambda, \lambda > 0$;
- (5) $(\lambda_1 + \lambda_2)M = \lambda_1 M \oplus \lambda_2 M, \forall \lambda_1, \lambda_2 > 0$;
- (6) $M^{\lambda_1} \otimes M^{\lambda_2} = M^{\lambda_1 + \lambda_2}, \forall \lambda_1, \lambda_2 > 0$.

It is easy to prove that the proposed PIVFHFES operation meets the requirements of Theorem 1. The operating rules provided by Theorem 1 hold significance in presenting aggregation operators detailed in subsequent sections, and they constitute a crucial foundation for this paper.

4. Probabilistic Interval-Valued Fermatean Hesitant Fuzzy Hamacher Aggregation Operators

Many operators are formulated based on the operations of Archimedes t-norm and t-conorm. The Hamacher operator, which is a special form of this, is introduced in this section. Two Hamacher weighted aggregation operators of PIVFHFESs are defined: the probabilistic interval-valued Fermatean hesitant fuzzy Hamacher weighted averaging (PIVFHFHWA) operator and geometric (PIVFHFHWG) operator. Their basic properties and special forms are discussed as well.

Definition 12. Let $M_i = \left(\left(\left[\mu_{M_{i_l}}^L(x), \mu_{M_{i_l}}^U(x) \right], \left[\nu_{M_{i_l}}^L(x), \nu_{M_{i_l}}^U(x) \right] \right), p_{i_l} \right)$ be a set of PIVFHFES, $i = 1, 2, \dots, n$, $l = 1, 2, \dots, L(\text{PIVFHFE})$. A probabilistic interval-valued Fermatean hesitant fuzzy Hamacher weighted averaging (PIVFHFHWA) operator is a function $\text{PIVFHFHWA} : \text{PIVFHFE}^n \rightarrow \text{PIVFHFE}$ such that

$$\text{PIVFHFHWA}(M_1, M_2, \dots, M_n) = \bigoplus_{i=1}^n (\omega_i M_i) \quad (18)$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weighting vector of M_i ($i = 1, 2, \dots, n$) with $0 \leq \omega_i \leq 1$ and $\sum_{i=1}^n \omega_i = 1$.

Theorem 2. Let $M_i = \left(\left(\left[\mu_{M_{i_l}}^L(x), \mu_{M_{i_l}}^U(x) \right], \left[\nu_{M_{i_l}}^L(x), \nu_{M_{i_l}}^U(x) \right] \right), p_{i_l} \right)$ be a collection of PIVFHFES, $i = 1, 2, \dots, n$, $l = 1, 2, \dots, L(\text{PIVFHFE})$. The aggregated result utilizing the PIVHFHWA operator is still a PIVFHFES, and

$$\text{PIVFHFHWA}(M_1, M_2, \dots, M_n) = \left(\left(\left[\frac{\prod_{i=1}^n \left(1 + (\delta - 1)(\mu_{M_{i_l}}^L)^3 \right)^{\omega_i} - \prod_{i=1}^n (1 - (\mu_{M_{i_l}}^L)^3)^{\omega_i}}{\prod_{i=1}^n \left(1 + (\delta - 1)(\mu_{M_{i_l}}^L)^3 \right)^{\omega_i} + (\delta - 1) \prod_{i=1}^n (1 - (\mu_{M_{i_l}}^L)^3)^{\omega_i}}, \frac{\prod_{i=1}^n \left(1 + (\delta - 1)(\mu_{M_{i_l}}^U)^3 \right)^{\omega_i} - \prod_{i=1}^n (1 - (\mu_{M_{i_l}}^U)^3)^{\omega_i}}{\prod_{i=1}^n \left(1 + (\delta - 1)(\mu_{M_{i_l}}^U)^3 \right)^{\omega_i} + (\delta - 1) \prod_{i=1}^n (1 - (\mu_{M_{i_l}}^U)^3)^{\omega_i}} \right], \left[\frac{\sqrt[3]{\delta} \prod_{i=1}^n (\nu_{M_{i_l}}^L)^{\omega_i}}{\sqrt[3]{\prod_{i=1}^n (1 + (\delta - 1)(1 - (\nu_{M_{i_l}}^L)^3))^{\omega_i} + (\delta - 1) \prod_{i=1}^n (\nu_{M_{i_l}}^L)^{3\omega_i}}}, \frac{\sqrt[3]{\delta} \prod_{i=1}^n (\nu_{M_{i_l}}^U)^{\omega_i}}{\sqrt[3]{\prod_{i=1}^n (1 + (\delta - 1)(1 - (\nu_{M_{i_l}}^U)^3))^{\omega_i} + (\delta - 1) \prod_{i=1}^n (\nu_{M_{i_l}}^U)^{3\omega_i}}} \right] \right), \overline{\sum_{i=1}^n p_{i_l}} \right) \quad (19)$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weighting vector of M_i with $0 \leq \omega_i \leq 1$ and $\sum_{i=1}^n \omega_i = 1$, and

$$\overline{\sum_{i=1}^n p_{i_l}} = \sum_{i=1}^n p_{i_l} / \sum_{l=1}^{L(\text{PIVFHFE})} \sum_{i=1}^n p_{i_l}.$$

Proof of Theorem 2.

When $n = 2$,

$$\begin{aligned}
 & PIVFHFHWA(M_1, M_2) = \\
 & \left(\left[\begin{array}{l} \sqrt[3]{\frac{(1+(\delta-1)(\mu_{M_1}^L)^3)^{\omega_1} - (1-(\mu_{M_1}^L)^3)^{\omega_1}}{(1+(\delta-1)(\mu_{M_1}^L)^3)^{\omega_1} + (\delta-1)(1-(\mu_{M_1}^L)^3)^{\omega_1}}} \\ \sqrt[3]{\frac{(1+(\delta-1)(\mu_{M_1}^U)^3)^{\omega_1} - (1-(\mu_{M_1}^U)^3)^{\omega_1}}{(1+(\delta-1)(\mu_{M_1}^U)^3)^{\omega_1} + (\delta-1)(1-(\mu_{M_1}^U)^3)^{\omega_1}}} \end{array} \right], \left[\begin{array}{l} \sqrt[3]{\frac{\sqrt[3]{\delta}(\nu_{M_1}^L)^{\omega_1}}{(1+(\delta-1)(1-(\nu_{M_1}^L)^3)^{\omega_1} + (\delta-1)(\nu_{M_1}^L)^{3\omega_1})}} \\ \sqrt[3]{\frac{\sqrt[3]{\delta}(\nu_{M_1}^U)^{\omega_1}}{(1+(\delta-1)(1-(\nu_{M_1}^U)^3)^{\omega_1} + (\delta-1)(\nu_{M_1}^U)^{3\omega_1})}} \end{array} \right], p_{1_l} \right) \\
 & \oplus \left(\left[\begin{array}{l} \sqrt[3]{\frac{(1+(\delta-1)(\mu_{M_2}^L)^3)^{\omega_2} - (1-(\mu_{M_2}^L)^3)^{\omega_2}}{(1+(\delta-1)(\mu_{M_2}^L)^3)^{\omega_2} + (\delta-1)(1-(\mu_{M_2}^L)^3)^{\omega_2}}} \\ \sqrt[3]{\frac{(1+(\delta-1)(\mu_{M_2}^U)^3)^{\omega_2} - (1-(\mu_{M_2}^U)^3)^{\omega_2}}{(1+(\delta-1)(\mu_{M_2}^U)^3)^{\omega_2} + (\delta-1)(1-(\mu_{M_2}^U)^3)^{\omega_2}}} \end{array} \right], \left[\begin{array}{l} \sqrt[3]{\frac{\sqrt[3]{\delta}(\nu_{M_2}^L)^{\omega_2}}{(1+(\delta-1)(1-(\nu_{M_2}^L)^3)^{\omega_2} + (\delta-1)(\nu_{M_2}^L)^{3\omega_2})}} \\ \sqrt[3]{\frac{\sqrt[3]{\delta}(\nu_{M_2}^U)^{\omega_2}}{(1+(\delta-1)(1-(\nu_{M_2}^U)^3)^{\omega_2} + (\delta-1)(\nu_{M_2}^U)^{3\omega_2})}} \end{array} \right], p_{2_k} \right) \\
 & = \left(\left[\begin{array}{l} \sqrt[3]{\frac{\prod_{i=1}^2 (1+(\delta-1)(\mu_{M_{i_l}}^L)^3)^{\omega_i} - \prod_{i=1}^2 (1-(\mu_{M_{i_l}}^L)^3)^{\omega_i}}{\prod_{i=1}^2 (1+(\delta-1)(\mu_{M_{i_l}}^L)^3)^{\omega_i} + (\delta-1)\prod_{i=1}^2 (1-(\mu_{M_{i_l}}^L)^3)^{\omega_i}}} \\ \sqrt[3]{\frac{\prod_{i=1}^2 (1+(\delta-1)(\mu_{M_{i_l}}^U)^3)^{\omega_i} - \prod_{i=1}^2 (1-(\mu_{M_{i_l}}^U)^3)^{\omega_i}}{\prod_{i=1}^2 (1+(\delta-1)(\mu_{M_{i_l}}^U)^3)^{\omega_i} + (\delta-1)\prod_{i=1}^2 (1-(\mu_{M_{i_l}}^U)^3)^{\omega_i}}} \end{array} \right], \left[\begin{array}{l} \sqrt[3]{\frac{\sqrt[3]{\delta} \prod_{i=1}^2 (\nu_{M_{i_l}}^L)^{\omega_i}}{\prod_{i=1}^2 (1+(\delta-1)(1-(\nu_{M_{i_l}}^L)^3)^{\omega_i} + (\delta-1)\prod_{i=1}^2 (\nu_{M_{i_l}}^L)^{3\omega_i})}} \\ \sqrt[3]{\frac{\sqrt[3]{\delta} \prod_{i=1}^2 (\nu_{M_{i_l}}^U)^{\omega_i}}{\prod_{i=1}^2 (1+(\delta-1)(1-(\nu_{M_{i_l}}^U)^3)^{\omega_i} + (\delta-1)\prod_{i=1}^2 (\nu_{M_{i_l}}^U)^{3\omega_i})}} \end{array} \right], \overbrace{\sum_{i=1}^2 p_{i_l}} \right)
 \end{aligned}$$

Suppose that Equation (19) holds for $n = s$, then

$$\begin{aligned}
 & PIVFHFHWA(M_1, M_2, \dots, M_s) = \\
 & \left(\left[\begin{array}{l} \sqrt[3]{\frac{\prod_{i=1}^s (1+(\delta-1)(\mu_{M_{i_l}}^L)^3)^{\omega_i} - \prod_{i=1}^s (1-(\mu_{M_{i_l}}^L)^3)^{\omega_i}}{\prod_{i=1}^s (1+(\delta-1)(\mu_{M_{i_l}}^L)^3)^{\omega_i} + (\delta-1)\prod_{i=1}^s (1-(\mu_{M_{i_l}}^L)^3)^{\omega_i}}} \\ \sqrt[3]{\frac{\prod_{i=1}^s (1+(\delta-1)(\mu_{M_{i_l}}^U)^3)^{\omega_i} - \prod_{i=1}^s (1-(\mu_{M_{i_l}}^U)^3)^{\omega_i}}{\prod_{i=1}^s (1+(\delta-1)(\mu_{M_{i_l}}^U)^3)^{\omega_i} + (\delta-1)\prod_{i=1}^s (1-(\mu_{M_{i_l}}^U)^3)^{\omega_i}}} \end{array} \right], \left[\begin{array}{l} \sqrt[3]{\frac{\sqrt[3]{\delta} \prod_{i=1}^s (\nu_{M_{i_l}}^L)^{\omega_i}}{\prod_{i=1}^s (1+(\delta-1)(1-(\nu_{M_{i_l}}^L)^3)^{\omega_i} + (\delta-1)\prod_{i=1}^s (\nu_{M_{i_l}}^L)^{3\omega_i})}} \\ \sqrt[3]{\frac{\sqrt[3]{\delta} \prod_{i=1}^s (\nu_{M_{i_l}}^U)^{\omega_i}}{\prod_{i=1}^s (1+(\delta-1)(1-(\nu_{M_{i_l}}^U)^3)^{\omega_i} + (\delta-1)\prod_{i=1}^s (\nu_{M_{i_l}}^U)^{3\omega_i})}} \end{array} \right], \overbrace{\sum_{i=1}^s p_{i_l}} \right)
 \end{aligned}$$

When $n = s + 1$,

$$\begin{aligned}
 & PIVFHFHWA(M_1, M_2, \dots, M_{s+1}) = PIVFHFHWA(M_1, M_2, \dots, M_s) \oplus (\omega_{s+1} M_{s+1}) = \\
 & \left(\left[\begin{array}{l} \sqrt[3]{\frac{\prod_{i=1}^s (1+(\delta-1)(\mu_{M_{i_l}}^L)^3)^{\omega_i} - \prod_{i=1}^s (1-(\mu_{M_{i_l}}^L)^3)^{\omega_i}}{\prod_{i=1}^s (1+(\delta-1)(\mu_{M_{i_l}}^L)^3)^{\omega_i} + (\delta-1)\prod_{i=1}^s (1-(\mu_{M_{i_l}}^L)^3)^{\omega_i}}} \\ \sqrt[3]{\frac{\prod_{i=1}^s (1+(\delta-1)(\mu_{M_{i_l}}^U)^3)^{\omega_i} - \prod_{i=1}^s (1-(\mu_{M_{i_l}}^U)^3)^{\omega_i}}{\prod_{i=1}^s (1+(\delta-1)(\mu_{M_{i_l}}^U)^3)^{\omega_i} + (\delta-1)\prod_{i=1}^s (1-(\mu_{M_{i_l}}^U)^3)^{\omega_i}}} \end{array} \right], \left[\begin{array}{l} \sqrt[3]{\frac{\sqrt[3]{\delta} \prod_{i=1}^s (\nu_{M_{i_l}}^L)^{\omega_i}}{\prod_{i=1}^s (1+(\delta-1)(1-(\nu_{M_{i_l}}^L)^3)^{\omega_i} + (\delta-1)\prod_{i=1}^s (\nu_{M_{i_l}}^L)^{3\omega_i})}} \\ \sqrt[3]{\frac{\sqrt[3]{\delta} \prod_{i=1}^s (\nu_{M_{i_l}}^U)^{\omega_i}}{\prod_{i=1}^s (1+(\delta-1)(1-(\nu_{M_{i_l}}^U)^3)^{\omega_i} + (\delta-1)\prod_{i=1}^s (\nu_{M_{i_l}}^U)^{3\omega_i})}} \end{array} \right], \overbrace{\sum_{i=1}^s p_{i_l}} \right) \\
 & \oplus \left(\left[\begin{array}{l} \sqrt[3]{\frac{(1+(\delta-1)(\mu_{M(s+1)_l}^L)^3)^{\omega_{s+1}} - (1-(\mu_{M(s+1)_l}^L)^3)^{\omega_{s+1}}}{(1+(\delta-1)(\mu_{M(s+1)_l}^L)^3)^{\omega_{s+1}} + (\delta-1)(1-(\mu_{M(s+1)_l}^L)^3)^{\omega_{s+1}}}} \\ \sqrt[3]{\frac{(1+(\delta-1)(\mu_{M(s+1)_l}^U)^3)^{\omega_{s+1}} - (1-(\mu_{M(s+1)_l}^U)^3)^{\omega_{s+1}}}{(1+(\delta-1)(\mu_{M(s+1)_l}^U)^3)^{\omega_{s+1}} + (\delta-1)(1-(\mu_{M(s+1)_l}^U)^3)^{\omega_{s+1}}}} \end{array} \right], \left[\begin{array}{l} \sqrt[3]{\frac{\sqrt[3]{\delta}(\nu_{M(s+1)_l}^L)^{\omega_{s+1}}}{(1+(\delta-1)(1-(\nu_{M(s+1)_l}^L)^3)^{\omega_{s+1}} + (\delta-1)(\nu_{M(s+1)_l}^L)^{3\omega_{s+1}})}} \\ \sqrt[3]{\frac{\sqrt[3]{\delta}(\nu_{M(s+1)_l}^U)^{\omega_{s+1}}}{(1+(\delta-1)(1-(\nu_{M(s+1)_l}^U)^3)^{\omega_{s+1}} + (\delta-1)(\nu_{M(s+1)_l}^U)^{3\omega_{s+1}})}} \end{array} \right], p_{s+1} \right) \\
 & = \left(\left[\begin{array}{l} \sqrt[3]{\frac{\prod_{i=1}^{s+1} (1+(\delta-1)(\mu_{M_{i_l}}^L)^3)^{\omega_i} - \prod_{i=1}^{s+1} (1-(\mu_{M_{i_l}}^L)^3)^{\omega_i}}{\prod_{i=1}^{s+1} (1+(\delta-1)(\mu_{M_{i_l}}^L)^3)^{\omega_i} + (\delta-1)\prod_{i=1}^{s+1} (1-(\mu_{M_{i_l}}^L)^3)^{\omega_i}}} \\ \sqrt[3]{\frac{\prod_{i=1}^{s+1} (1+(\delta-1)(\mu_{M_{i_l}}^U)^3)^{\omega_i} - \prod_{i=1}^{s+1} (1-(\mu_{M_{i_l}}^U)^3)^{\omega_i}}{\prod_{i=1}^{s+1} (1+(\delta-1)(\mu_{M_{i_l}}^U)^3)^{\omega_i} + (\delta-1)\prod_{i=1}^{s+1} (1-(\mu_{M_{i_l}}^U)^3)^{\omega_i}}} \end{array} \right], \left[\begin{array}{l} \sqrt[3]{\frac{\sqrt[3]{\delta} \prod_{i=1}^{s+1} (\nu_{M_{i_l}}^L)^{\omega_i}}{\prod_{i=1}^{s+1} (1+(\delta-1)(1-(\nu_{M_{i_l}}^L)^3)^{\omega_i} + (\delta-1)\prod_{i=1}^{s+1} (\nu_{M_{i_l}}^L)^{3\omega_i})}} \\ \sqrt[3]{\frac{\sqrt[3]{\delta} \prod_{i=1}^{s+1} (\nu_{M_{i_l}}^U)^{\omega_i}}{\prod_{i=1}^{s+1} (1+(\delta-1)(1-(\nu_{M_{i_l}}^U)^3)^{\omega_i} + (\delta-1)\prod_{i=1}^{s+1} (\nu_{M_{i_l}}^U)^{3\omega_i})}} \end{array} \right], \overbrace{\sum_{i=1}^{s+1} p_{i_l}} \right)
 \end{aligned}$$

So, when $n = s + 1$, Equation (19) is true. $\square$

Example 2. Let $M_1 = \langle ([0.7, 0.9], [0.4, 0.5], 0.6), ([0.5, 0.5], [0.3, 0.7], 0.4) \rangle$ and $M_2 = \langle ([0.3, 0.4], [0.2, 0.5], 0.5), ([0.6, 0.8], [0.5, 0.7], 0.5) \rangle$ be two PIVFHEs and $\omega = (0.7, 0.3)^T$ be the weighting vector of $M_i (i = 1, 2)$. Assume that $\delta = 1.5$, then calculate PIVFHHWA (M_1, M_2) by Equation (19).

$$\begin{aligned} \overline{M}_1 &= \left\langle \left[\begin{array}{l} \sqrt[3]{\frac{(1+(1.5-1) \times 0.7^3)^{0.7} \times (1+(1.5-1) \times 0.3^3)^{0.3} - ((1-0.7^3)^{0.7} \times (1-0.3^3)^{0.3})}{(1+(1.5-1) \times 0.7^3)^{0.7} \times (1+(1.5-1) \times 0.3^3)^{0.3} + (1.5-1)(1-0.7^3)^{0.7}(1-0.3^3)^{0.3}}} \\ \sqrt[3]{\frac{(1+(1.5-1) \times 0.9^3)^{0.7} \times (1+(1.5-1) \times 0.2^3)^{0.3} - ((1-0.9^3)^{0.7} \times (1-0.2^3)^{0.3})}{(1+(1.5-1) \times 0.9^3)^{0.7} \times (1+(1.5-1) \times 0.2^3)^{0.3} + (1.5-1)(1-0.9^3)^{0.7}(1-0.2^3)^{0.3}}} \\ \sqrt[3]{1.5 \times 0.4^{0.7} \times 0.2^{0.3}} \\ \sqrt[3]{(1+(1.5-1)(1-0.4^3))^{0.7} \times (1+(1.5-1)(1-0.2^3))^{0.7} + (1.5-1) \times 0.4^{0.7} \times 0.2^{0.3}} \\ \sqrt[3]{1.5 \times 0.5^{0.7} \times 0.5^{0.3}} \\ \sqrt[3]{(1+(1.5-1)(1-0.5^3))^{0.7} \times (1+(1.5-1)(1-0.5^3))^{0.7} + (1.5-1) \times 0.5^{0.7} \times 0.5^{0.3}} \end{array} \right], \overline{p}_1 \right\rangle \\ &= \langle ([0.6354, 0.8405], [0.3254, 0.5000]), 0.550 \rangle \end{aligned}$$

$$\text{where } \overline{p}_1 = \frac{0.6+0.5}{0.6+0.4+0.5+0.5} = \frac{1.1}{2} = 0.550.$$

$$\begin{aligned} \overline{M}_2 &= \left\langle \left[\begin{array}{l} \sqrt[3]{\frac{(1+(1.5-1) \times 0.5^3)^{0.7} \times (1+(1.5-1) \times 0.6^3)^{0.3} - ((1-0.5^3)^{0.7} \times (1-0.6^3)^{0.3})}{(1+(1.5-1) \times 0.5^3)^{0.7} \times (1+(1.5-1) \times 0.6^3)^{0.3} + (1.5-1)(1-0.5^3)^{0.7}(1-0.6^3)^{0.3}}} \\ \sqrt[3]{\frac{(1+(1.5-1) \times 0.5^3)^{0.7} \times (1+(1.5-1) \times 0.8^3)^{0.3} - ((1-0.5^3)^{0.7} \times (1-0.8^3)^{0.3})}{(1+(1.5-1) \times 0.5^3)^{0.7} \times (1+(1.5-1) \times 0.8^3)^{0.3} + (1.5-1)(1-0.5^3)^{0.7}(1-0.8^3)^{0.3}}} \\ \sqrt[3]{1.5 \times 0.3^{0.7} \times 0.5^{0.3}} \\ \sqrt[3]{(1+(1.5-1)(1-0.3^3))^{0.7} \times (1+(1.5-1)(1-0.5^3))^{0.7} + (1.5-1) \times 0.3^{0.7} \times 0.5^{0.3}} \\ \sqrt[3]{1.5 \times 0.7^{0.7} \times 0.7^{0.3}} \\ \sqrt[3]{(1+(1.5-1)(1-0.7^3))^{0.7} \times (1+(1.5-1)(1-0.7^3))^{0.7} + (1.5-1) \times 0.7^{0.7} \times 0.7^{0.3}} \end{array} \right], \overline{p}_2 \right\rangle \\ &= \langle ([0.5348, 0.6395], [0.3502, 0.7000]), 0.450 \rangle \end{aligned}$$

$$\text{where } \overline{p}_2 = \frac{0.5+0.4}{0.6+0.4+0.5+0.5} = \frac{0.9}{2} = 0.450. \text{ Therefore, we have}$$

$$PIVFHHWA(M_1, M_2) = \left\langle \begin{array}{l} ([0.6534, 0.8405], [0.3254, 0.5000], 0.550), \\ ([0.5348, 0.6365], [0.3502, 0.7000], 0.450) \end{array} \right\rangle$$

Theorem 3. (Boundness) Let M^+ and M^- be two PIVFHEs, and $M^+ = \left[\left(\left(\mu_{Mi_l}^L \right)^+, \left(\mu_{Mi_l}^U \right)^+ \right), \left[\left(\left(\mu_{Mi_l}^L \right)^-, \left(\mu_{Mi_l}^U \right)^- \right), p_{i_l} \right], M^- = \left[\left(\left(\mu_{Mi_l}^L \right)^-, \left(\mu_{Mi_l}^U \right)^- \right), \left[\left(\left(\mu_{Mi_l}^L \right)^+, \left(\mu_{Mi_l}^U \right)^+ \right), p_{i_l} \right] \right]$ where $i = 1, 2, \dots, n, l = 1, 2, \dots, L(\text{PIVFHFE})$. If $(\mu_{Mi_l}^L)^+ = \max \{ \mu_{Mi_l}^L \}, (\mu_{Mi_l}^U)^+ = \max \{ \mu_{Mi_l}^U \}, (\mu_{Mi_l}^L)^- = \min \{ \mu_{Mi_l}^L \}, (\mu_{Mi_l}^U)^- = \min \{ \mu_{Mi_l}^U \}, (\mu_{Mi_l}^L)^+ = \max \{ \mu_{Mi_l}^L \}, (\mu_{Mi_l}^U)^+ = \max \{ \mu_{Mi_l}^U \}, (\mu_{Mi_l}^L)^- = \min \{ \mu_{Mi_l}^L \}, (\mu_{Mi_l}^U)^- = \min \{ \mu_{Mi_l}^U \}, (\mu_{Mi_l}^L)^- = \min \{ \mu_{Mi_l}^L \}$ and $(\mu_{Mi_l}^U)^- = \min \{ \mu_{Mi_l}^U \}$, then

$$M^- \leq \text{PIVFHHWA}(M_1, M_2, \dots, M_n) \leq M^+ \quad (20)$$

Theorem 4. (Monotonicity) Let $M_i = \left[\left(\left(\mu_{Mi_l}^L, \mu_{Mi_l}^U \right), \left[\left(\mu_{Mi_l}^L, \mu_{Mi_l}^U \right), p_{i_l} \right] \right) \right]$ and $\tilde{M}_i = \left[\left(\left(\mu_{Mi_l}^L, \mu_{Mi_l}^U \right), \left[\left(\mu_{Mi_l}^L, \mu_{Mi_l}^U \right), p_{i_l} \right] \right) \right]$ be two set of PIVFHEs, where $i = 1, 2, \dots, n, l = 1, 2, \dots, L(\text{PIVFHFE})$. If $\mu_{Mi_l}^L \leq \mu_{\tilde{M}_i}^L, \mu_{Mi_l}^U \leq \mu_{\tilde{M}_i}^U, \mu_{Mi_l}^L \geq \mu_{\tilde{M}_i}^L$ and $\mu_{Mi_l}^U \geq \mu_{\tilde{M}_i}^U$, then

$$\text{PIVFHHWA}(M_1, M_2, \dots, M_n) \leq \text{PIVFHHWA}(\tilde{M}_1, \tilde{M}_2, \dots, \tilde{M}_n) \quad (21)$$

Theorem 5. (Idempotency) Let $M_i = \left(\left(\left[\mu_{M_i}^L, \mu_{M_i}^U \right], \left[v_{M_i}^L, v_{M_i}^U \right] \right), p_i \right)$ be a set of PIVFHFES, where $i = 1, 2, \dots, n, l = 1, 2, \dots, L(\text{PIVFHFE})$. If $M_i = M = \left(\left(\left[\mu_{M_i}^L, \mu_{M_i}^U \right], \left[v_{M_i}^L, v_{M_i}^U \right] \right), p_i \right)$, then

$$\text{PIVFHFHWA}(M_1, M_2, \dots, M_n) = M \quad (22)$$

Some special forms of the PIVHFHWA operator regarding different parameter δ are given as follows.

When $\delta = 1$, Equation (19) follows that

$$\text{PIVFHFHWA}(M_1, M_2, \dots, M_n) = \left\langle \left[\sqrt[3]{1 - \prod_{i=1}^n \left(1 - \left(\mu_{M_i}^L \right)^3 \right)^{\omega_i}}, \sqrt[3]{1 - \prod_{i=1}^n \left(1 - \left(\mu_{M_i}^U \right)^3 \right)^{\omega_i}} \right], \left[\prod_{i=1}^n \left(v_{M_i}^L \right)^{3\omega_i}, \prod_{i=1}^n \left(v_{M_i}^U \right)^{3\omega_i} \right], \overline{\sum_{i=1}^n p_i} \right\rangle \quad (23)$$

The PIVHFHWA operator reduces to probabilistic interval-valued Fermatean hesitant fuzzy weighted averaging (PIVFHFWA) operator.

When $\delta = 2$, it follows that

$$\text{PIVFHFHWA}(M_1, M_2, \dots, M_n) = \left\langle \left[\sqrt[3]{\frac{\prod_{i=1}^n \left(1 + \left(\mu_{M_i}^L \right)^3 \right)^{\omega_i} - \prod_{i=1}^n \left(1 - \left(\mu_{M_i}^L \right)^3 \right)^{\omega_i}}{\prod_{i=1}^n \left(1 + \left(\mu_{M_i}^L \right)^3 \right)^{\omega_i} + \prod_{i=1}^n \left(1 - \left(\mu_{M_i}^L \right)^3 \right)^{\omega_i}}}, \sqrt[3]{\frac{\prod_{i=1}^n \left(1 + \left(\mu_{M_i}^U \right)^3 \right)^{\omega_i} - \prod_{i=1}^n \left(1 - \left(\mu_{M_i}^U \right)^3 \right)^{\omega_i}}{\prod_{i=1}^n \left(1 + \left(\mu_{M_i}^U \right)^3 \right)^{\omega_i} + \prod_{i=1}^n \left(1 - \left(\mu_{M_i}^U \right)^3 \right)^{\omega_i}}} \right], \left[\sqrt[3]{\frac{\prod_{i=1}^n \left(1 + \left(v_{M_i}^L \right)^3 \right)^{\omega_i} - \prod_{i=1}^n \left(1 - \left(v_{M_i}^L \right)^3 \right)^{\omega_i}}{\prod_{i=1}^n \left(1 + \left(v_{M_i}^L \right)^3 \right)^{\omega_i} + \prod_{i=1}^n \left(1 - \left(v_{M_i}^L \right)^3 \right)^{\omega_i}}}, \sqrt[3]{\frac{\prod_{i=1}^n \left(1 + \left(v_{M_i}^U \right)^3 \right)^{\omega_i} - \prod_{i=1}^n \left(1 - \left(v_{M_i}^U \right)^3 \right)^{\omega_i}}{\prod_{i=1}^n \left(1 + \left(v_{M_i}^U \right)^3 \right)^{\omega_i} + \prod_{i=1}^n \left(1 - \left(v_{M_i}^U \right)^3 \right)^{\omega_i}}} \right], \overline{\sum_{i=1}^n p_i} \right\rangle \quad (24)$$

The PIVHFHWA operator reduces to a probabilistic interval-valued Fermatean hesitant fuzzy Einstein weighted averaging (PIVFHFHWA) operator.

Definition 13. Let $M_i = \left(\left(\left[\mu_{M_i}^L(x), \mu_{M_i}^U(x) \right], \left[v_{M_i}^L(x), v_{M_i}^U(x) \right] \right), p_i \right)$ be a set of PIVFHFES, $i = 1, 2, \dots, n, l = 1, 2, \dots, L(\text{PIVFHFE})$. A probabilistic interval-valued Fermatean fuzzy Hamacher weighted geometric (PIVFHFHWG) operator is a function $\text{PIVFHFHWG} : \text{PIVFHFE}^n \rightarrow \text{PIVFHFE}$ such that

$$\text{PIVFHFHWG}(M_1, M_2, \dots, M_n) = \bigotimes_{i=1}^n M_i^{\omega_i} \quad (25)$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weighting vector of $M_i (i = 1, 2, \dots, n)$ with $0 \leq \omega_i \leq 1$ and $\sum_{i=1}^n \omega_i = 1$.

Theorem 6. Let $M_i = \left(\left(\left[\mu_{M_i}^L(x), \mu_{M_i}^U(x) \right], \left[v_{M_i}^L(x), v_{M_i}^U(x) \right] \right), p_i \right)$ be a set of PIVFHFES, $i = 1, 2, \dots, n, l = 1, 2, \dots, L(\text{PIVFHFE})$. The aggregated result obtained by using PIVHFHWG operator integration is still a PIVFHFES, and

$$\text{PIVFHFHWG}(M_1, M_2, \dots, M_n) = \bigotimes_{i=1}^n M_i^{\omega_i} = M_1^{\omega_1} \otimes M_2^{\omega_2} \otimes \dots \otimes M_n^{\omega_n} = \left\langle \left[\sqrt[3]{\frac{\prod_{i=1}^n \left(1 + (\delta-1) \left(\mu_{M_i}^L \right)^3 \right)^{\omega_i} - \prod_{i=1}^n \left(1 - (\delta-1) \left(\mu_{M_i}^L \right)^3 \right)^{\omega_i}}{\prod_{i=1}^n \left(1 + (\delta-1) \left(\mu_{M_i}^L \right)^3 \right)^{\omega_i} + \prod_{i=1}^n \left(1 - (\delta-1) \left(\mu_{M_i}^L \right)^3 \right)^{\omega_i}}}, \sqrt[3]{\frac{\prod_{i=1}^n \left(1 + (\delta-1) \left(\mu_{M_i}^U \right)^3 \right)^{\omega_i} - \prod_{i=1}^n \left(1 - (\delta-1) \left(\mu_{M_i}^U \right)^3 \right)^{\omega_i}}{\prod_{i=1}^n \left(1 + (\delta-1) \left(\mu_{M_i}^U \right)^3 \right)^{\omega_i} + \prod_{i=1}^n \left(1 - (\delta-1) \left(\mu_{M_i}^U \right)^3 \right)^{\omega_i}}} \right], \left[\sqrt[3]{\frac{\prod_{i=1}^n \left(1 + (\delta-1) \left(v_{M_i}^L \right)^3 \right)^{\omega_i} - \prod_{i=1}^n \left(1 - (\delta-1) \left(v_{M_i}^L \right)^3 \right)^{\omega_i}}{\prod_{i=1}^n \left(1 + (\delta-1) \left(v_{M_i}^L \right)^3 \right)^{\omega_i} + \prod_{i=1}^n \left(1 - (\delta-1) \left(v_{M_i}^L \right)^3 \right)^{\omega_i}}}, \sqrt[3]{\frac{\prod_{i=1}^n \left(1 + (\delta-1) \left(v_{M_i}^U \right)^3 \right)^{\omega_i} - \prod_{i=1}^n \left(1 - (\delta-1) \left(v_{M_i}^U \right)^3 \right)^{\omega_i}}{\prod_{i=1}^n \left(1 + (\delta-1) \left(v_{M_i}^U \right)^3 \right)^{\omega_i} + \prod_{i=1}^n \left(1 - (\delta-1) \left(v_{M_i}^U \right)^3 \right)^{\omega_i}}} \right], \overline{\sum_{i=1}^n p_i} \right\rangle \quad (26)$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weighting vector with $0 \leq \omega_i \leq 1$ and $\sum_{i=1}^n \omega_i = 1$, and

$$\sum_{i=1}^n p_{i_l} = \sum_{i=1}^n p_{i_l} / \sum_{l=1}^{L(\text{PIVFHFE})} \sum_{i=1}^n p_{i_l}.$$

The PIVFHFHWG operator also has some excellent properties. Please refer to Theorems 3–5. When $\delta = 1$, Equation (26) reduces to a probabilistic interval-valued Fermatean hesitant fuzzy weighted geometric (PIVFHFWG) operator; when $\delta = 2$, it reduces to a probabilistic interval-valued Fermatean hesitant fuzzy Einstein weighted geometric (PIVFHFEWG) operator.

5. Hamacher Choquet Integral Aggregation Operators of PIVFHSs

In reality, the attributes involved in the decision-making process are not always independent of each other and often have certain correlations. Therefore, considering the correlations between different attributes and the individual preferences of decision makers, the PIVFHFHCIA and PIVFHFHCIG operators are introduced and their excellent properties are discussed.

Definition 14. Let X be a fixed set. $M_i = \left[\left(\left[\mu_{M_{i_l}}^L(x), \mu_{M_{i_l}}^U(x) \right], \left[\nu_{M_{i_l}}^L(x), \nu_{M_{i_l}}^U(x) \right] \right), p_{i_l} \right]$ is a set of PIVFHFES on X with $i = 1, 2, \dots, n$ and $l = 1, 2, \dots, L(\text{PIVFHFE})$. Then, the PIVFHFHCIA operator is given as:

$$\begin{aligned} & \text{PIVFHFHCIA}(M_1, M_2, \dots, M_n) \\ &= \bigoplus_{i=1}^n \left(\left(\kappa(\rho(i)) - \kappa(\rho(i+1)) \right) M_i \right) \\ &= \left(\left(\kappa(\rho(1)) - \kappa(\rho(2)) \right) M_1 \right) \oplus \left(\left(\kappa(\rho(2)) - \kappa(\rho(3)) \right) M_2 \right) \oplus \dots \oplus \left(\left(\kappa(\rho(n)) - \kappa(\rho(n+1)) \right) M_n \right) \end{aligned} \quad (27)$$

Theorem 7. Let $M_i = \left[\left(\left[\mu_{M_{i_l}}^L(x), \mu_{M_{i_l}}^U(x) \right], \left[\nu_{M_{i_l}}^L(x), \nu_{M_{i_l}}^U(x) \right] \right), p_{i_l} \right]$ be a set of PIVFHFES on a finite set X with $i = 1, 2, \dots, n$ and $l = 1, 2, \dots, L(\text{PIVFHFE})$. κ represents the fuzzy measure, then the aggregation result utilizing PIVFHFHCIG operator is still a PIVFHFEN.

$$\begin{aligned} & \text{PIVFHFHCIA}(M_1, M_2, \dots, M_n) \\ &= \left[\begin{aligned} & \sqrt[3]{\frac{\prod_{i=1}^n \left(1 + (\delta - 1) \left(\mu_{M_{i_l}}^L \right)^3 \right)^{\kappa(\rho(i)) - \kappa(\rho(i+1))} - \prod_{i=1}^n \left(1 - \left(\mu_{M_{i_l}}^L \right)^3 \right)^{\kappa(\rho(i)) - \kappa(\rho(i+1))}}{\prod_{i=1}^n \left(1 + (\delta - 1) \left(\mu_{M_{i_l}}^L \right)^3 \right)^{\kappa(\rho(i)) - \kappa(\rho(i+1))} + (\delta - 1) \prod_{i=1}^n \left(1 - \left(\mu_{M_{i_l}}^L \right)^3 \right)^{\kappa(\rho(i)) - \kappa(\rho(i+1))}}} \right)^{\kappa(\rho(i)) - \kappa(\rho(i+1))}} \\ & \sqrt[3]{\frac{\prod_{i=1}^n \left(1 + (\delta - 1) \left(\mu_{M_{i_l}}^U \right)^3 \right)^{\kappa(\rho(i)) - \kappa(\rho(i+1))} - \prod_{i=1}^n \left(1 - \left(\mu_{M_{i_l}}^U \right)^3 \right)^{\kappa(\rho(i)) - \kappa(\rho(i+1))}}{\prod_{i=1}^n \left(1 + (\delta - 1) \left(\mu_{M_{i_l}}^U \right)^3 \right)^{\kappa(\rho(i)) - \kappa(\rho(i+1))} + (\delta - 1) \prod_{i=1}^n \left(1 - \left(\mu_{M_{i_l}}^U \right)^3 \right)^{\kappa(\rho(i)) - \kappa(\rho(i+1))}}} \right)^{\kappa(\rho(i)) - \kappa(\rho(i+1))}} \\ & \sqrt[3]{\frac{\prod_{i=1}^n \left(1 + (\delta - 1) \left(\nu_{M_{i_l}}^L \right)^3 \right)^{\kappa(\rho(i)) - \kappa(\rho(i+1))} - \prod_{i=1}^n \left(1 - \left(\nu_{M_{i_l}}^L \right)^3 \right)^{\kappa(\rho(i)) - \kappa(\rho(i+1))}}{\prod_{i=1}^n \left(1 + (\delta - 1) \left(\nu_{M_{i_l}}^L \right)^3 \right)^{\kappa(\rho(i)) - \kappa(\rho(i+1))} + (\delta - 1) \prod_{i=1}^n \left(1 - \left(\nu_{M_{i_l}}^L \right)^3 \right)^{\kappa(\rho(i)) - \kappa(\rho(i+1))}}} \right)^{\kappa(\rho(i)) - \kappa(\rho(i+1))}} \\ & \sqrt[3]{\frac{\prod_{i=1}^n \left(1 + (\delta - 1) \left(\nu_{M_{i_l}}^U \right)^3 \right)^{\kappa(\rho(i)) - \kappa(\rho(i+1))} - \prod_{i=1}^n \left(1 - \left(\nu_{M_{i_l}}^U \right)^3 \right)^{\kappa(\rho(i)) - \kappa(\rho(i+1))}}{\prod_{i=1}^n \left(1 + (\delta - 1) \left(\nu_{M_{i_l}}^U \right)^3 \right)^{\kappa(\rho(i)) - \kappa(\rho(i+1))} + (\delta - 1) \prod_{i=1}^n \left(1 - \left(\nu_{M_{i_l}}^U \right)^3 \right)^{\kappa(\rho(i)) - \kappa(\rho(i+1))}}} \right)^{\kappa(\rho(i)) - \kappa(\rho(i+1))}} \end{aligned} \right], \sqrt[n]{\sum_{i=1}^n p_{i_l}} \end{aligned} \quad (28)$$

$$\text{where } \sum_{i=1}^n p_{i_l} = \sum_{i=1}^n p_{i_l} / \sum_{l=1}^{L(\text{PIVFHFE})} \sum_{i=1}^n p_{i_l}.$$

Proof of Theorem 7. When $n = 2$, we have

$$\begin{aligned}
 PIVFHFHCIA(M_1, M_2) &= \kappa_{1,2} M_1 \oplus \kappa_{2,3} M_2 = \\
 &= \left\langle \left[\begin{array}{c} \sqrt[3]{\frac{(1+(\delta-1)(\mu_{M_1}^L)^3)^{\kappa_{1,2}} - (1-(\mu_{M_1}^L)^3)^{\kappa_{1,2}}}{(1+(\delta-1)(\mu_{M_1}^L)^3)^{\kappa_{1,2}} + (\delta-1)(1-(\mu_{M_1}^L)^3)^{\kappa_{1,2}}}} \\ \sqrt[3]{\frac{(1+(\delta-1)(\mu_{M_1}^U)^3)^{\kappa_{1,2}} - (1-(\mu_{M_1}^U)^3)^{\kappa_{1,2}}}{(1+(\delta-1)(\mu_{M_1}^U)^3)^{\kappa_{1,2}} + (\delta-1)(1-(\mu_{M_1}^U)^3)^{\kappa_{1,2}}}} \end{array} \right], \left[\begin{array}{c} \sqrt[3]{\frac{\sqrt[3]{\delta} (v_{M_1}^L)^{\kappa_{1,2}}}{(1+(\delta-1)(1-(v_{M_1}^L)^3)^{\kappa_{1,2}} + (\delta-1)(v_{M_1}^L)^{3\kappa_{1,2}})}} \\ \sqrt[3]{\frac{\sqrt[3]{\delta} (v_{M_1}^U)^{\kappa_{1,2}}}{(1+(\delta-1)(1-(v_{M_1}^U)^3)^{\kappa_{1,2}} + (\delta-1)(v_{M_1}^U)^{3\kappa_{1,2}}}}} \end{array} \right], p_{1_l} \right\rangle \\
 &\oplus \left\langle \left[\begin{array}{c} \sqrt[3]{\frac{(1+(\delta-1)(\mu_{M_2}^L)^3)^{\kappa_{2,3}} - (1-(\mu_{M_2}^L)^3)^{\kappa_{2,3}}}{(1+(\delta-1)(\mu_{M_2}^L)^3)^{\kappa_{2,3}} + (\delta-1)(1-(\mu_{M_2}^L)^3)^{\kappa_{2,3}}}} \\ \sqrt[3]{\frac{(1+(\delta-1)(\mu_{M_2}^U)^3)^{\kappa_{2,3}} - (1-(\mu_{M_2}^U)^3)^{\kappa_{2,3}}}{(1+(\delta-1)(\mu_{M_2}^U)^3)^{\kappa_{2,3}} + (\delta-1)(1-(\mu_{M_2}^U)^3)^{\kappa_{2,3}}}} \end{array} \right], \left[\begin{array}{c} \sqrt[3]{\frac{\sqrt[3]{\delta} (v_{M_2}^L)^{\kappa_{2,3}}}{(1+(\delta-1)(1-(v_{M_2}^L)^3)^{\kappa_{2,3}} + (\delta-1)(v_{M_2}^L)^{3\kappa_{2,3}})}} \\ \sqrt[3]{\frac{\sqrt[3]{\delta} (v_{M_2}^U)^{\kappa_{2,3}}}{(1+(\delta-1)(1-(v_{M_2}^U)^3)^{\kappa_{2,3}} + (\delta-1)(v_{M_2}^U)^{3\kappa_{2,3}}}}} \end{array} \right], p_{2_k} \right\rangle \\
 &= \left\langle \left[\begin{array}{c} \sqrt[3]{\frac{\prod_{i=1}^2 (1+(\delta-1)(\mu_{M_{i_l}}^L)^3)^{\kappa_{i,j+1}} - \prod_{i=1}^2 (1-(\mu_{M_{i_l}}^L)^3)^{\kappa_{i,j+1}}}{\prod_{i=1}^2 (1+(\delta-1)(\mu_{M_{i_l}}^L)^3)^{\kappa_{i,j+1}} + (\delta-1) \prod_{i=1}^2 (1-(\mu_{M_{i_l}}^L)^3)^{\kappa_{i,j+1}}}} \\ \sqrt[3]{\frac{\prod_{i=1}^2 (1+(\delta-1)(\mu_{M_{i_l}}^U)^3)^{\kappa_{i,j+1}} - \prod_{i=1}^2 (1-(\mu_{M_{i_l}}^U)^3)^{\kappa_{i,j+1}}}{\prod_{i=1}^2 (1+(\delta-1)(\mu_{M_{i_l}}^U)^3)^{\kappa_{i,j+1}} + (\delta-1) \prod_{i=1}^2 (1-(\mu_{M_{i_l}}^U)^3)^{\kappa_{i,j+1}}}} \end{array} \right], \left[\begin{array}{c} \sqrt[3]{\frac{\sqrt[3]{\delta} \prod_{i=1}^2 (v_{M_{i_l}}^L)^{\kappa_{i,j+1}}}{\prod_{i=1}^2 (1+(\delta-1)(1-(v_{M_{i_l}}^L)^3)^{\kappa_{i,j+1}} + (\delta-1) \prod_{i=1}^2 (v_{M_{i_l}}^L)^{3\kappa_{i,j+1}})}} \\ \sqrt[3]{\frac{\sqrt[3]{\delta} \prod_{i=1}^2 (v_{M_{i_l}}^U)^{\kappa_{i,j+1}}}{\prod_{i=1}^2 (1+(\delta-1)(1-(v_{M_{i_l}}^U)^3)^{\kappa_{i,j+1}} + (\delta-1) \prod_{i=1}^2 (v_{M_{i_l}}^U)^{3\kappa_{i,j+1}}}}} \end{array} \right], \overbrace{\sum_{i=1}^2 p_{i_l}} \right\rangle
 \end{aligned}$$

where $i = 1, 2, \dots, l, l = 1, 2, \dots, L(PIVFHFCE_1), k = 1, 2, \dots, L(PIVFHFCE_2)$.

Assume that when $n = s$, the theorem is true.

$$\begin{aligned}
 PIVFHFHCIA(M_1, M_2, \dots, M_n) \\
 &= \left\langle \left[\begin{array}{c} \sqrt[3]{\frac{\prod_{i=1}^s (1+(\delta-1)(\mu_{M_{i_l}}^L)^3)^{\kappa_{i,j+1}} - \prod_{i=1}^s (1-(\mu_{M_{i_l}}^L)^3)^{\kappa_{i,j+1}}}{\prod_{i=1}^s (1+(\delta-1)(\mu_{M_{i_l}}^L)^3)^{\kappa_{i,j+1}} + (\delta-1) \prod_{i=1}^s (1-(\mu_{M_{i_l}}^L)^3)^{\kappa_{i,j+1}}}} \\ \sqrt[3]{\frac{\prod_{i=1}^s (1+(\delta-1)(\mu_{M_{i_l}}^U)^3)^{\kappa_{i,j+1}} - \prod_{i=1}^s (1-(\mu_{M_{i_l}}^U)^3)^{\kappa_{i,j+1}}}{\prod_{i=1}^s (1+(\delta-1)(\mu_{M_{i_l}}^U)^3)^{\kappa_{i,j+1}} + (\delta-1) \prod_{i=1}^s (1-(\mu_{M_{i_l}}^U)^3)^{\kappa_{i,j+1}}}} \end{array} \right], \left[\begin{array}{c} \sqrt[3]{\frac{\sqrt[3]{\delta} \prod_{i=1}^s (v_{M_{i_l}}^L)^{\kappa_{i,j+1}}}{\prod_{i=1}^s (1+(\delta-1)(1-(v_{M_{i_l}}^L)^3)^{\kappa_{i,j+1}} + (\delta-1) \prod_{i=1}^s (v_{M_{i_l}}^L)^{3\kappa_{i,j+1}})}} \\ \sqrt[3]{\frac{\sqrt[3]{\delta} \prod_{i=1}^s (v_{M_{i_l}}^U)^{\kappa_{i,j+1}}}{\prod_{i=1}^s (1+(\delta-1)(1-(v_{M_{i_l}}^U)^3)^{\kappa_{i,j+1}} + (\delta-1) \prod_{i=1}^s (v_{M_{i_l}}^U)^{3\kappa_{i,j+1}}}}} \end{array} \right], \overbrace{\sum_{i=1}^s p_{i_l}} \right\rangle
 \end{aligned}$$

Then, when $n = s + 1$, we have

$$\begin{aligned}
 PIVFHFHCIA(M_1, M_2, \dots, M_n) &= PIVFHFHCIA(M_1, M_2, \dots, M_s) \oplus \kappa_{s,s+1} M_{s+1} = \\
 &= \left\langle \left[\begin{array}{c} \sqrt[3]{\frac{\prod_{i=1}^s (1+(\delta-1)(\mu_{M_{i_l}}^L)^3)^{\kappa_{i,j+1}} - \prod_{i=1}^s (1-(\mu_{M_{i_l}}^L)^3)^{\kappa_{i,j+1}}}{\prod_{i=1}^s (1+(\delta-1)(\mu_{M_{i_l}}^L)^3)^{\kappa_{i,j+1}} + (\delta-1) \prod_{i=1}^s (1-(\mu_{M_{i_l}}^L)^3)^{\kappa_{i,j+1}}}} \\ \sqrt[3]{\frac{\prod_{i=1}^s (1+(\delta-1)(\mu_{M_{i_l}}^U)^3)^{\kappa_{i,j+1}} - \prod_{i=1}^s (1-(\mu_{M_{i_l}}^U)^3)^{\kappa_{i,j+1}}}{\prod_{i=1}^s (1+(\delta-1)(\mu_{M_{i_l}}^U)^3)^{\kappa_{i,j+1}} + (\delta-1) \prod_{i=1}^s (1-(\mu_{M_{i_l}}^U)^3)^{\kappa_{i,j+1}}}} \end{array} \right], \left[\begin{array}{c} \sqrt[3]{\frac{\sqrt[3]{\delta} \prod_{i=1}^s (v_{M_{i_l}}^L)^{\kappa_{i,j+1}}}{\prod_{i=1}^s (1+(\delta-1)(1-(v_{M_{i_l}}^L)^3)^{\kappa_{i,j+1}} + (\delta-1) \prod_{i=1}^s (v_{M_{i_l}}^L)^{3\kappa_{i,j+1}})}} \\ \sqrt[3]{\frac{\sqrt[3]{\delta} \prod_{i=1}^s (v_{M_{i_l}}^U)^{\kappa_{i,j+1}}}{\prod_{i=1}^s (1+(\delta-1)(1-(v_{M_{i_l}}^U)^3)^{\kappa_{i,j+1}} + (\delta-1) \prod_{i=1}^s (v_{M_{i_l}}^U)^{3\kappa_{i,j+1}}}}} \end{array} \right], \overbrace{\sum_{i=1}^s p_{i_l}} \right\rangle \\
 &\oplus \left\langle \left[\begin{array}{c} \sqrt[3]{\frac{(1+(\delta-1)(\mu_{M(s+1)_l}^L)^3)^{\kappa_{s+1,s+2}} - (1-(\mu_{M(s+1)_l}^L)^3)^{\kappa_{s+1,s+2}}}{(1+(\delta-1)(\mu_{M(s+1)_l}^L)^3)^{\kappa_{s+1,s+2}} + (\delta-1)(1-(\mu_{M(s+1)_l}^L)^3)^{\kappa_{s+1,s+2}}}} \\ \sqrt[3]{\frac{(1+(\delta-1)(\mu_{M(s+1)_l}^U)^3)^{\kappa_{s+1,s+2}} - (1-(\mu_{M(s+1)_l}^U)^3)^{\kappa_{s+1,s+2}}}{(1+(\delta-1)(\mu_{M(s+1)_l}^U)^3)^{\kappa_{s+1,s+2}} + (\delta-1)(1-(\mu_{M(s+1)_l}^U)^3)^{\kappa_{s+1,s+2}}}} \end{array} \right], \left[\begin{array}{c} \sqrt[3]{\frac{\sqrt[3]{\delta} (v_{M(s+1)_l}^L)^{\kappa_{s+1,s+2}}}{(1+(\delta-1)(1-(v_{M(s+1)_l}^L)^3)^{\kappa_{s+1,s+2}} + (\delta-1)(v_{M(s+1)_l}^L)^{3\kappa_{s+1,s+2}}}}} \\ \sqrt[3]{\frac{\sqrt[3]{\delta} (v_{M(s+1)_l}^U)^{\kappa_{s+1,s+2}}}{(1+(\delta-1)(1-(v_{M(s+1)_l}^U)^3)^{\kappa_{s+1,s+2}} + (\delta-1)(v_{M(s+1)_l}^U)^{3\kappa_{s+1,s+2}}}}} \end{array} \right], p_{s+1} \right\rangle \\
 &= \left\langle \left[\begin{array}{c} \sqrt[3]{\frac{\prod_{i=1}^{s+1} (1+(\delta-1)(\mu_{M_{i_l}}^L)^3)^{\kappa_{i,j+1}} - \prod_{i=1}^{s+1} (1-(\mu_{M_{i_l}}^L)^3)^{\kappa_{i,j+1}}}{\prod_{i=1}^{s+1} (1+(\delta-1)(\mu_{M_{i_l}}^L)^3)^{\kappa_{i,j+1}} + (\delta-1) \prod_{i=1}^{s+1} (1-(\mu_{M_{i_l}}^L)^3)^{\kappa_{i,j+1}}}} \\ \sqrt[3]{\frac{\prod_{i=1}^{s+1} (1+(\delta-1)(\mu_{M_{i_l}}^U)^3)^{\kappa_{i,j+1}} - \prod_{i=1}^{s+1} (1-(\mu_{M_{i_l}}^U)^3)^{\kappa_{i,j+1}}}{\prod_{i=1}^{s+1} (1+(\delta-1)(\mu_{M_{i_l}}^U)^3)^{\kappa_{i,j+1}} + (\delta-1) \prod_{i=1}^{s+1} (1-(\mu_{M_{i_l}}^U)^3)^{\kappa_{i,j+1}}}} \end{array} \right], \left[\begin{array}{c} \sqrt[3]{\frac{\sqrt[3]{\delta} \prod_{i=1}^{s+1} (v_{M_{i_l}}^L)^{\kappa_{i,j+1}}}{\prod_{i=1}^{s+1} (1+(\delta-1)(1-(v_{M_{i_l}}^L)^3)^{\kappa_{i,j+1}} + (\delta-1) \prod_{i=1}^{s+1} (v_{M_{i_l}}^L)^{3\kappa_{i,j+1}})}} \\ \sqrt[3]{\frac{\sqrt[3]{\delta} \prod_{i=1}^{s+1} (v_{M_{i_l}}^U)^{\kappa_{i,j+1}}}{\prod_{i=1}^{s+1} (1+(\delta-1)(1-(v_{M_{i_l}}^U)^3)^{\kappa_{i,j+1}} + (\delta-1) \prod_{i=1}^{s+1} (v_{M_{i_l}}^U)^{3\kappa_{i,j+1}}}}} \end{array} \right], \overbrace{\sum_{i=1}^{s+1} p_{i_l}} \right\rangle
 \end{aligned}$$

Therefore, when $n = s + 1$, Theorem 7 is true. $\square$

Example 3. Let $M_1 = ([0.2, 0.4], [0.3, 0.6], 0.4)$, $([0.5, 0.8], [0.4, 0.7], 0.6)$, $M_2 = ([0.3, 0.6], [0.1, 0.2], 0.2)$, $([0.4, 0.5], [0.3, 0.6], 0.8)$, and $M_3 = ([0.5, 0.7], [0.2, 0.4], 0.5)$, $([0.4, 0.6], [0.7, 0.9], 0.4)$ be three PIVFHEs. If the fuzzy measure is: $\kappa(m_1) = 0.35$, $\kappa(m_2) = 0.25$, $\kappa(m_3) = 0.30$, then we have $\gamma = \prod_{i=1}^3 (1 + \gamma\kappa(m_i)) - 1 = (1 + 0.35\gamma)(1 + 0.25\gamma)(1 + 0.30\gamma) - 1$. Solve this equation, $\gamma = 0.36$. Using Equation (12), we can obtain:

$$\kappa(m_1, m_2) = \frac{1}{0.36} [(1 + 0.36 \times 0.35)(1 + 0.36 \times 0.25) - 1] = 0.63$$

$$\kappa(m_1, m_3) = \frac{1}{0.36} [(1 + 0.36 \times 0.35)(1 + 0.36 \times 0.30) - 1] = 0.69$$

$$\kappa(m_2, m_3) = \frac{1}{0.36} [(1 + 0.36 \times 0.25)(1 + 0.36 \times 0.30) - 1] = 0.58$$

$$\kappa(m_1, m_2, m_3) = \frac{1}{0.36} [(1 + 0.36 \times 0.35)(1 + 0.36 \times 0.25)(1 + 0.36 \times 0.30) - 1] = 1$$

Using Equation (26) to solve the Choquet integral, then

$$\kappa(M_1, M_2) = \kappa(m_1, m_2, m_3) - \kappa(m_2, m_3) = 1 - 0.58 = 0.42$$

$$\kappa(M_2, M_3) = \kappa(m_2, m_3) - \kappa(m_3) = 0.58 - 0.30 = 0.28$$

$$\kappa(M_3, M_4) = \kappa(m_3, m_4) - \kappa(m_4) = \kappa(m_3) = 0.30$$

Suppose $\delta = 1.5$, we can obtain the following results based on the PIVFHFHCIA operator:

$$\begin{aligned} \widetilde{M}_1 &= \left\langle \left[\begin{aligned} &\sqrt[3]{\frac{(1+(1.5-1) \times 0.2^3)^{0.42} \times (1+(1.5-3) \times 0.3^3)^{0.28} \times (1+(1.5-3) \times 0.5^3)^{0.30} - (1-0.2^3)^{0.42} (1-0.3^3)^{0.28} (1-0.5^3)^{0.30}}{(1+(1.5-1) \times 0.2^3)^{0.42} \times (1+(1.5-3) \times 0.3^3)^{0.28} \times (1+(1.5-3) \times 0.5^3)^{0.30} + (1.5-1)(1-0.2^3)^{0.42} (1-0.3^3)^{0.28} (1-0.5^3)^{0.30}}} \right.} \\ &\sqrt[3]{\frac{(1+(1.5-1) \times 0.4^3)^{0.42} \times (1+(1.5-3) \times 0.6^3)^{0.28} \times (1+(1.5-3) \times 0.7^3)^{0.30} - (1-0.4^3)^{0.42} (1-0.6^3)^{0.28} (1-0.7^3)^{0.30}}{(1+(1.5-1) \times 0.4^3)^{0.42} \times (1+(1.5-3) \times 0.6^3)^{0.28} \times (1+(1.5-3) \times 0.7^3)^{0.30} + (1.5-1)(1-0.4^3)^{0.42} (1-0.6^3)^{0.28} (1-0.7^3)^{0.30}}} \\ &\frac{\sqrt[3]{1.5 \times 0.3^{0.42} \times 0.1^{0.28} \times 0.2^{0.30}}}{\sqrt[3]{(1+(1.5-1)(1-0.3^3))^{0.42} (1+(1.5-1)(1-0.1^3))^{0.28} (1+(1.5-1)(1-0.2^3))^{0.30} + (1.5-1) \times 0.3^3 \times 0.42 \times 0.1^3 \times 0.28 \times 0.2^3 \times 0.30}}} \\ &\frac{\sqrt[3]{1.5 \times 0.6^{0.42} \times 0.2^{0.28} \times 0.4^{0.30}}}{\sqrt[3]{(1+(1.5-1)(1-0.6^3))^{0.42} (1+(1.5-1)(1-0.2^3))^{0.28} (1+(1.5-1)(1-0.4^3))^{0.30} + (1.5-1) \times 0.6^3 \times 0.42 \times 0.2^3 \times 0.28 \times 0.4^3 \times 0.30}}} \end{aligned} \right], \widetilde{p}_1 \right\rangle \\ &= ([0.3664, 0.5807], [0.1954, 0.3930], 0.3793) \end{aligned}$$

$$\text{where } \widetilde{p}_1 = \frac{0.4+0.2+0.5}{0.4+0.6+0.2+0.8+0.5+0.4} = \frac{1.1}{2.9} = 0.3793.$$

$$\begin{aligned} \widetilde{M}_2 &= \left\langle \left[\begin{aligned} &\sqrt[3]{\frac{(1+(1.5-1) \times 0.5^3)^{0.42} \times (1+(1.5-3) \times 0.4^3)^{0.28} \times (1+(1.5-3) \times 0.4^3)^{0.30} - (1-0.5^3)^{0.42} (1-0.4^3)^{0.28} (1-0.4^3)^{0.30}}{(1+(1.5-1) \times 0.5^3)^{0.42} \times (1+(1.5-3) \times 0.4^3)^{0.28} \times (1+(1.5-3) \times 0.4^3)^{0.30} + (1.5-1)(1-0.5^3)^{0.42} (1-0.4^3)^{0.28} (1-0.4^3)^{0.30}}} \right.} \\ &\sqrt[3]{\frac{(1+(1.5-1) \times 0.8^3)^{0.42} \times (1+(1.5-3) \times 0.5^3)^{0.28} \times (1+(1.5-3) \times 0.6^3)^{0.30} - (1-0.8^3)^{0.42} (1-0.5^3)^{0.28} (1-0.6^3)^{0.30}}{(1+(1.5-1) \times 0.8^3)^{0.42} \times (1+(1.5-3) \times 0.5^3)^{0.28} \times (1+(1.5-3) \times 0.6^3)^{0.30} + (1.5-1)(1-0.8^3)^{0.42} (1-0.5^3)^{0.28} (1-0.6^3)^{0.30}}} \\ &\frac{\sqrt[3]{1.5 \times 0.4^{0.42} \times 0.3^{0.28} \times 0.7^{0.30}}}{\sqrt[3]{(1+(1.5-1)(1-0.4^3))^{0.42} (1+(1.5-1)(1-0.3^3))^{0.28} (1+(1.5-1)(1-0.7^3))^{0.30} + (1.5-1) \times 0.4^3 \times 0.42 \times 0.3^3 \times 0.28 \times 0.7^3 \times 0.30}}} \\ &\frac{\sqrt[3]{1.5 \times 0.7^{0.42} \times 0.6^{0.28} \times 0.9^{0.30}}}{\sqrt[3]{(1+(1.5-1)(1-0.7^3))^{0.42} (1+(1.5-1)(1-0.6^3))^{0.28} (1+(1.5-1)(1-0.9^3))^{0.30} + (1.5-1) \times 0.7^3 \times 0.42 \times 0.6^3 \times 0.28 \times 0.9^3 \times 0.30}}} \end{aligned} \right], \widetilde{p}_2 \right\rangle \\ &= ([0.4480, 0.6918], [0.4393, 0.7273], 0.6209) \end{aligned}$$

$$\text{where } \widetilde{p}_2 = \frac{0.6+0.8+0.4}{0.4+0.6+0.2+0.8+0.5+0.4} = \frac{1.8}{2.9} = 0.6207. \text{ Then,}$$

$$PIVFHFHCIA(M_1, M_2, M_3) = \left\langle \begin{matrix} ([0.3664, 0.5807], [0.1954, 0.3930], 0.3793), \\ ([0.4480, 0.6918], [0.4393, 0.7273], 0.6207) \end{matrix} \right\rangle.$$

Theorem 8. (Boundness) Let M^+ and M^- be two PIVFHFES, and $M^+ = \left[\left(\left[(\mu_{M_i}^L)^+, (\mu_{M_i}^U)^+ \right], \left[(v_{M_i}^L)^-, (v_{M_i}^U)^- \right] \right), p_i \right]$, $M^- = \left[\left(\left[(\mu_{M_i}^L)^-, (\mu_{M_i}^U)^- \right], \left[(v_{M_i}^L)^+, (v_{M_i}^U)^+ \right] \right), p_i \right]$ with $i = 1, 2, \dots, n$ and $l = 1, 2, \dots, L(PIVFHFE)$. If $(\mu_{M_i}^L)^+ = \max\{\mu_{M_i}^L\}$, $(\mu_{M_i}^U)^+ = \max\{\mu_{M_i}^U\}$, $(v_{M_i}^L)^+ = \max\{v_{M_i}^L\}$, $(v_{M_i}^U)^+ = \max\{v_{M_i}^U\}$, $(\mu_{M_i}^L)^- = \min\{\mu_{M_i}^L\}$, $(\mu_{M_i}^U)^- = \min\{\mu_{M_i}^U\}$, $(v_{M_i}^L)^- = \min\{v_{M_i}^L\}$ and $(v_{M_i}^U)^- = \min\{v_{M_i}^U\}$, then

$$M^- \leq PIVFHFHCIA(M_1, M_2, \dots, M_n) \leq M^+ \quad (29)$$

Theorem 9. (Monotonicity) Let $M_i = \left[\left(\left[\mu_{M_i}^L, \mu_{M_i}^U \right], \left[v_{M_i}^L, v_{M_i}^U \right] \right), p_i \right]$ and $\tilde{M}_i = \left[\left(\left[\mu_{\tilde{M}_i}^L, \mu_{\tilde{M}_i}^U \right], \left[v_{\tilde{M}_i}^L, v_{\tilde{M}_i}^U \right] \right), p_i \right]$ be two set of PIVFHFES with $i = 1, 2, \dots, n$ and $l = 1, 2, \dots, L(PIVFHFE)$. If $\mu_{M_i}^L \leq \mu_{\tilde{M}_i}^L$, $\mu_{M_i}^U \leq \mu_{\tilde{M}_i}^U$, $v_{M_i}^L \geq v_{\tilde{M}_i}^L$ and $v_{M_i}^U \geq v_{\tilde{M}_i}^U$, then

$$PIVFHFHCIA(M_1, M_2, \dots, M_n) \leq PIVFHFHCIA(\tilde{M}_1, \tilde{M}_2, \dots, \tilde{M}_n) \quad (30)$$

Theorem 10. (Permutation invariance) Let $\ddot{M}_i = (\ddot{M}_1, \ddot{M}_2, \dots, \ddot{M}_n)$ be any permutation of $M_i = (M_1, M_2, \dots, M_n)$, then

$$PIVFHFHCIA(\ddot{M}_1, \ddot{M}_2, \dots, \ddot{M}_n) = PIVFHFHCIA(M_1, M_2, \dots, M_n) \quad (31)$$

Theorem 11. (Idempotency) Let $M_i = \left[\left(\left[\mu_{M_i}^L, \mu_{M_i}^U \right], \left[v_{M_i}^L, v_{M_i}^U \right] \right), p_i \right]$ be a PIVFHFES with $i = 1, 2, \dots, n$ and $l = 1, 2, \dots, L(PIVFHFE)$. If $M_i = M = \left[\left(\left[\mu_M^L, \mu_M^U \right], \left[v_M^L, v_M^U \right] \right), p \right]$, then

$$PIVFHFHCIA(M_1, M_2, \dots, M_n) = M \quad (32)$$

Definition 15. Let X be a fixed set. $M_i = \left[\left(\left[\mu_{M_i}^L, \mu_{M_i}^U \right], \left[v_{M_i}^L, v_{M_i}^U \right] \right), p_i \right]$ is a set of PIVFHFES on X with $i = 1, 2, \dots, n$ and $l = 1, 2, \dots, L(PIVFHFE)$. Then, the PIVFHFHCIG operator is an object of the following form:

$$\begin{aligned} & PIVFHFHCIG(M_1, M_2, \dots, M_n) \\ &= \bigotimes_{i=1}^n M_i^{\kappa(\rho(i)) - \kappa(\rho(i+1))} \\ &= M_1^{\kappa(\rho(1)) - \kappa(\rho(2))} \otimes M_2^{\kappa(\rho(2)) - \kappa(\rho(3))} \otimes \dots \otimes M_n^{\kappa(\rho(n)) - \kappa(\rho(n+1))} \end{aligned} \quad (33)$$

where (i) is a permutation of M_i , which satisfies the condition $M_{(1)} \leq M_{(2)} \leq \dots \leq M_{(n)}$. κ is the fuzzy measure, $\rho(i) = \{\Lambda(i), \Lambda(i+1), \dots, \Lambda(n)\}$ and $\rho(n+1) = 0$.

Theorem 12. Let $M_i = \left[\left(\left[\mu_{M_i}^L(x), \mu_{M_i}^U(x) \right], \left[\nu_{M_i}^L(x), \nu_{M_i}^U(x) \right] \right), p_i \right]$ be a set of PIVFHFES on a finite set X with $i = 1, 2, \dots, n$ and $l = 1, 2, \dots, L(\text{PIVFHFE})$. κ represents the fuzzy measure, then the aggregation result utilizing PIVFHFHCIG operator is still a PIVFHFN.

$$\text{PIVFHFHCIG}(M_1, M_2, \dots, M_n) = \left(\left[\begin{aligned} & \sqrt[n]{\delta \prod_{i=1}^n (\mu_{M_i}^L)^{\kappa(\rho(i)) - \kappa(\rho(i+1))}} \\ & \sqrt[n]{\prod_{i=1}^n (1 + (\delta - 1)(1 - (\mu_{M_i}^L)^3))^{\kappa(\rho(i)) - \kappa(\rho(i+1))} + (\delta - 1) \prod_{i=1}^n (\mu_{M_i}^L)^{3\kappa(\rho(i)) - \kappa(\rho(i+1))}} \\ & \sqrt[n]{\delta \prod_{i=1}^n (\mu_{M_i}^U)^{\kappa(\rho(i)) - \kappa(\rho(i+1))}} \\ & \sqrt[n]{\prod_{i=1}^n (1 + (\delta - 1)(1 - (\mu_{M_i}^U)^3))^{\kappa(\rho(i)) - \kappa(\rho(i+1))} + (\delta - 1) \prod_{i=1}^n (\mu_{M_i}^U)^{\kappa(\rho(i)) - \kappa(\rho(i+1))}} \\ & \sqrt[n]{\prod_{i=1}^n (1 + (\delta - 1)(\nu_{M_i}^L)^3)^{\kappa(\rho(i)) - \kappa(\rho(i+1))} - \prod_{i=1}^n (1 - (\nu_{M_i}^L)^3)^{\kappa(\rho(i)) - \kappa(\rho(i+1))}} \\ & \sqrt[n]{\prod_{i=1}^n (1 + (\delta - 1)(\nu_{M_i}^L)^3)^{\kappa(\rho(i)) - \kappa(\rho(i+1))} + (\delta - 1) \prod_{i=1}^n (1 - (\nu_{M_i}^L)^3)^{\kappa(\rho(i)) - \kappa(\rho(i+1))}} \\ & \sqrt[n]{\prod_{i=1}^n (1 + (\delta - 1)(\nu_{M_i}^U)^3)^{\kappa(\rho(i)) - \kappa(\rho(i+1))} - \prod_{i=1}^n (1 - (\nu_{M_i}^U)^3)^{\kappa(\rho(i)) - \kappa(\rho(i+1))}} \\ & \sqrt[n]{\prod_{i=1}^n (1 + (\delta - 1)(\nu_{M_i}^U)^3)^{\kappa(\rho(i)) - \kappa(\rho(i+1))} + (\delta - 1) \prod_{i=1}^n (1 - (\nu_{M_i}^U)^3)^{\kappa(\rho(i)) - \kappa(\rho(i+1))}} \end{aligned} \right], \sum_{i=1}^n p_i \right] \quad (34)$$

where $\sum_{i=1}^n p_i = \sum_{i=1}^n p_i / \sum_{l=1}^{L(\text{PIVFHFE})} \sum_{i=1}^n p_i$.

Similarly, PIVFHFHCIG operator has some of the same excellent properties as the PIVFHFHCIA operator. Please refer to Theorem 8–11 above.

6. MADM Method Based on PIVFHFHCIG Operator

A MADM method for solving problems with PIVFHFES information is proposed, and the effectiveness and rationality of the proposed method are verified through a numerical example of carbon emission reduction of manufacturers in the supply chain, which provides a significant reference for the development of FFSs.

6.1. Decision-Making Method

For a MADM problem, let $Y_i = \{Y_1, Y_2, \dots, Y_n\}$ be a collection of alternatives and $C_i = \{C_1, C_2, \dots, C_n\}$ be a collection of attributes. Assume that the alternative Y_i with respect to the attribute C_i is expressed by a PIVFHFES $M_{ij}(i = 1, 2, \dots, m, j = 1, 2, \dots, n)$. To fully depict the differences between evaluating opinions, a probabilistic interval-valued Fermatean hesitant fuzzy decision matrix $D = (M_{ij})_{m \times n}$ using PIVFHFES is constructed to express the opinions of experts regarding different alternative Y_i with respect to attribute C_i . Then, the decision matrix is formed by using the opinion from decision makers. It is assumed that all attribute information is known here, so the involved fuzzy measures are directly given by experts according to their knowledge and experience. A new decision method based on PIVFHFHCIG operator for solving MADM problems is proposed in this section and illustrated by the following steps:

Step 1. Establish the decision matrix of PIVFHFES $D = (M_{ij})_{m \times n}$ according to the evaluation information provided by the experts.

Step 2. Normalize the decision matrix $D = (M_{ij})_{m \times n}$. Then,

$$\tilde{M}_{ij} = \begin{cases} M_{ij}, & \text{when } C_j \text{ is a benefit-type attribute} \\ \overline{M}_{ij}, & \text{when } C_j \text{ is a cost-type attribute} \end{cases}$$

where $M_{ij} = ([\mu_{ij}^L, \mu_{ij}^U], [\nu_{ij}^L, \nu_{ij}^U]), p_{ij}]$ and $\overline{M}_{ij} = ([\nu_{ij}^L, \nu_{ij}^U], [\mu_{ij}^L, \mu_{ij}^U]), p_{ij}]$.

Step 3. The fuzzy measure κ_i of each attribute is directly given by the experts according to the practical significance and decision requirements. After that, calculate the value of parameter γ and Choquet integrals by Definition 6 and 7.

$$\kappa(\Lambda) = \begin{cases} \frac{1}{\gamma} \left(\prod_{i=1}^n (1 + \gamma \kappa(\Lambda_i)) - 1 \right), & \gamma \neq 0 \\ \sum_{i=1}^n \kappa(\Lambda_i), & \gamma = 0 \end{cases} \quad (35)$$

$$\gamma = \prod_{i=1}^n (1 + \gamma \kappa(\Lambda_i)) - 1 \quad (36)$$

$$\Lambda_{\kappa}(f(\Lambda_{(1)}), f(\Lambda_{(2)}), \dots, f(\Lambda_{(n)})) = \sum_{i=1}^n f(\Lambda_{(i)}) [\kappa(\rho_{(i)}) - \kappa(\rho_{(i+1)})] \quad (37)$$

Step 4. Use the PIVFHFHCIG operator in Equation (34) to calculate the comprehensive values of each alternative.

Step 5. Use the following scoring function and accuracy function to obtain the final scores of alternatives.

$$S(M_i) = \sum_{l=1}^{L(PIVFHFE)} p_{i_l} \times \frac{\left((\mu_{M_{i_l}}^L(x))^3 - (v_{M_{i_l}}^U(x))^3 \right) + \left((\mu_{M_{i_l}}^U(x))^3 - (v_{M_{i_l}}^L(x))^3 \right)}{2} \quad (38)$$

$$E(M_i) = \sum_{l=1}^{L(PIVFHFE)} p_{i_l} \times \frac{2 - \left((\mu_{M_{i_l}}^L(x))^3 - (\pi_{M_{i_l}}^U(x))^3 \right)}{2} \quad (39)$$

Step 6. The alternatives are ranked by the comparison rules of PIVHPFEs in Definition 10, and then the best scheme can be selected.

6.2. Numerical Example

As global climate warming and energy security issues become increasingly serious, green and low-carbon development has become a consensus of the international community. To address the global climate crisis that threatens human survival and reduce carbon dioxide emissions, many countries have established targets for reducing carbon emissions, developed and enforced carbon trading standards to combat climate change, and encouraged the transition of their economic models to a low-carbon, green approach. Under the background of energy conservation and emission reduction, supply chain enterprises need to develop green and low-carbon development strategies to cope with regulatory pressure and satisfy growing consumer demands for eco-friendly options. By doing so, they will enhance their own profits, optimize resource utilization, and effectively reduce carbon emissions. For manufacturers in the supply chain, there is pressure to develop low-carbon technologies and implement green and clean production actively.

Under the background of low-carbon economy, assume that Enterprise W needs to evaluate the low-carbon development of the four manufacturers and select a suitable enterprise from them for cooperation to further promote carbon emission reduction and enhance the green level of the supply chain. W has now organized four experts to conduct a preliminary evaluation of the indicators of the four manufacturers, which are: C_1 : Green technology innovation; C_2 : Green product development; C_3 : Resource utilization rate; C_4 : Pollutant discharge. The alternatives are assessed by an evaluation panel composed of several experts, and the information during the process is expressed as PIVFHFes. For example, when the panel hesitates between several possible interval values when evaluating the first manufacturer regarding attribute C_1 . They are 50% sure that the green technology innovation of Y_1 is likely to be from 4 to 5 but not less than 6 to 8 and 50% sure that the green technology innovation of Y_1 could be from 4 to 5 but not less than 5 to 6 (10 points are full marks). After data normalization, the opinion can be stated as $\langle ([0.4, 0.5], [0.6, 0.8], 0.5), ([0.4, 0.5], [0.5, 0.6], 0.5) \rangle$. Therefore, the probabilistic interval-

valued Fermatean hesitant fuzzy decision matrix $D = (M_{ij})_{4 \times 4}$ is constructed through the discussion of experts.

Step 1. Establish the decision matrix $D = (M_{ij})_{m \times n}$ based on the evaluation information provided by experts, as shown in Table 1.

Table 1. The decision matrix of PIVFHFNs.

	C_1	C_2	C_3	C_4
Y_1	$\langle ([0.4, 0.5], [0.6, 0.8], 0.5), ([0.4, 0.5], [0.5, 0.6], 0.5) \rangle$	$\langle ([0.1, 0.3], [0.4, 0.6], 0.3), ([0.2, 0.4], [0.3, 0.5], 0.7) \rangle$	$\langle ([0.3, 0.5], [0.7, 0.9], 0.6), ([0.2, 0.4], [0.4, 0.6], 0.3) \rangle$	$\langle ([0.3, 0.5], [0.4, 0.5], 0.4), ([0.3, 0.4], [0.2, 0.4], 0.6) \rangle$
Y_2	$\langle ([0.2, 0.4], [0.3, 0.5], 0.3), ([0.3, 0.5], [0.4, 0.6], 0.7) \rangle$	$\langle ([0.3, 0.5], [0.6, 0.8], 0.5), ([0.3, 0.5], [0.5, 0.6], 0.4) \rangle$	$\langle ([0.5, 0.7], [0.6, 0.8], 0.3), ([0.2, 0.4], [0.3, 0.5], 0.6) \rangle$	$\langle ([0.1, 0.4], [0.3, 0.6], 0.8), ([0.2, 0.5], [0.1, 0.4], 0.2) \rangle$
Y_3	$\langle ([0.5, 0.7], [0.4, 0.8], 0.4), ([0.4, 0.6], [0.4, 0.5], 0.6) \rangle$	$\langle ([0.2, 0.5], [0.4, 0.7], 0.5), ([0.3, 0.4], [0.4, 0.5], 0.5) \rangle$	$\langle ([0.7, 0.9], [0.3, 0.6], 0.3), ([0.2, 0.3], [0.3, 0.5], 0.7) \rangle$	$\langle ([0.4, 0.6], [0.2, 0.3], 0.45), ([0.3, 0.4], [0.5, 0.6], 0.5) \rangle$
Y_4	$\langle ([0.3, 0.5], [0.1, 0.4], 0.4), ([0.4, 0.5], [0.3, 0.6], 0.6) \rangle$	$\langle ([0.4, 0.7], [0.6, 0.8], 0.2), ([0.3, 0.4], [0.2, 0.5], 0.8) \rangle$	$\langle ([0.2, 0.4], [0.4, 0.5], 0.4), ([0.3, 0.5], [0.5, 0.6], 0.6) \rangle$	$\langle ([0.5, 0.6], [0.3, 0.4], 0.7), ([0.2, 0.4], [0.5, 0.7], 0.3) \rangle$

Step 2. The decision matrix does not to be standardized since $\{Y_1, Y_2, \dots, Y_4\}$ are all benefit-type attributes.

Step 3. When identifying fuzzy measures, decision makers can directly give the values of κ_i according to the practical meaning of each attribute and decision requirements. In this paper, the fuzzy measures are obtained according to the opinions of experts: $\kappa(c_1) = 0.4$, $\kappa(c_2) = 0.3$, $\kappa(c_3) = 0.1$, $\kappa(c_4) = 0.4$. Then, we can obtain an equation about the parameter γ by Equation (36):

$$\gamma = \prod_{i=1}^n (1 + \gamma \kappa(c_i)) = (1 + 0.4\gamma)(1 + 0.3\gamma)(1 + 0.1\gamma)(1 + 0.4\gamma) - 1$$

The value of γ is obtained using MATLAB: $\gamma = -0.4222$. Then, we have

$$\begin{aligned} \kappa(c_1, c_2) &= \frac{1}{\gamma} \left(\prod_{i=1}^n (1 + \gamma \kappa(c_i)) - 1 \right) \\ &= \frac{1}{-0.4222} [(1 - 0.4222 \times 0.4)(1 - 0.4222 \times 0.3) - 1] = 0.649 \end{aligned}$$

Similarly, we can obtain $\kappa(c_1, c_3) = 0.483$, $\kappa(c_1, c_4) = 0.732$, $\kappa(c_2, c_3) = 0.387$, $\kappa(c_2, c_4) = 0.649$, $\kappa(c_3, c_4) = 0.483$, $\kappa(c_1, c_2, c_3) = 0.722$, $\kappa(c_1, c_2, c_4) = 0.940$, $\kappa(c_1, c_3, c_4) = 0.802$, $\kappa(c_2, c_3, c_4) = 0.722$, $\kappa(c_1, c_2, c_3, c_4) = 1.000$.

Then, the Choquet integral is calculated by Definition 7.

$$\kappa(Y_1 - Y_2) = \kappa(c_1, c_2, c_3, c_4) - \kappa(c_2, c_3, c_4) = 1 - 0.722 = 0.278$$

$$\kappa(Y_2 - Y_3) = \kappa(c_2, c_3, c_4) - \kappa(c_3, c_4) = 0.722 - 0.483 = 0.239$$

$$\kappa(Y_3 - Y_4) = \kappa(c_3, c_4) - \kappa(c_4) = 0.483 - 0.4 = 0.083$$

$$\kappa(Y_4 - Y_5) = \kappa(c_4) - \kappa(c_5) = \kappa(c_4) = 0.4$$

Step 4. Use the PIVFHFHCIG operator in Equation (34) to calculate the comprehensive scores of each alternative (when $\delta = 0.8$).

$$M_1 = \left\langle ([0.2496, 0.4420], [0.5139, 0.7211], 0.4615), ([0.2850, 0.4255], [0.3715, 0.5149], 0.5385) \right\rangle$$

$$M_2 = \left\langle ([0.1780, 0.4414], [0.4534, 0.6762], 0.5000), ([0.2466, 0.4908], [0.3729, 0.5328], 0.5000) \right\rangle$$

$$M_3 = \left\langle \begin{array}{l} ([0.3766, 0.6185], [0.3390, 0.6590], 0.4177) \\ ([0.3141, 0.4365], [0.4415, 0.5460], 0.5823) \end{array} \right\rangle$$

$$M_4 = \left\langle \begin{array}{l} ([0.3806, 0.5712], [0.4178, 0.5925], 0.4250) \\ ([0.2761, 0.4334], [0.4153, 0.6307], 0.5750) \end{array} \right\rangle$$

Step 5. Calculate final scores of each alternative Y_i by the following scoring function:

$$S(M_i) = \sum_{l=1}^{L(PIVFHFE)} p_{i_l} \times \frac{\left(\left(\mu_{M_{i_l}}^L(x) \right)^3 - \left(\nu_{M_{i_l}}^U(x) \right)^3 \right) + \left(\left(\mu_{M_{i_l}}^U(x) \right)^3 - \left(\nu_{M_{i_l}}^L(x) \right)^3 \right)}{2}$$

we can have

$$S(Y_1) = -0.1179, S(Y_2) = -0.0951, S(Y_3) = -0.0466, S(Y_4) = -0.0717.$$

Step 6. According to the calculation result above and relevant comparison rules in Definition 10, the enterprises are ranked as $Y_3 \succ Y_4 \succ Y_2 \succ Y_1$. That is, Y_3 is the best scheme, and Enterprise W should select Y_3 after considering all factors.

6.3. Sensitivity Analysis

To further validate the scientificity of the operators proposed in this paper, the effects of different parameter values on the aggregation results of PIVFHFHCIG operators are discussed further. When parameter δ selects values of 0.5, 1, 2, 3, 5, 6.5, 8, 10, 15, and 20, the final scores and rankings are displayed in Table 2.

Table 2. The scores and ranking of the alternatives when parameter δ selects different values.

δ	$S(Y_1)$	$S(Y_2)$	$S(Y_3)$	$S(Y_4)$	Ranking
0.5	−0.1208	−0.0974	−0.0497	−0.0750	$Y_3 \succ Y_4 \succ Y_2 \succ Y_1$
0.8	−0.1179	−0.0951	−0.0466	−0.0771	$Y_3 \succ Y_4 \succ Y_2 \succ Y_1$
1	−0.1164	−0.0938	−0.0449	−0.0699	$Y_3 \succ Y_4 \succ Y_2 \succ Y_1$
2	−0.1114	−0.0892	−0.0394	−0.0637	$Y_3 \succ Y_4 \succ Y_2 \succ Y_1$
3	−0.1084	−0.0861	−0.0359	−0.0597	$Y_3 \succ Y_4 \succ Y_2 \succ Y_1$
5	−0.1045	−0.0820	−0.0313	−0.0545	$Y_3 \succ Y_4 \succ Y_2 \succ Y_1$
6.5	−0.1027	−0.0798	−0.0290	−0.0519	$Y_3 \succ Y_4 \succ Y_2 \succ Y_1$
8	−0.1013	−0.0782	−0.0272	−0.0499	$Y_3 \succ Y_4 \succ Y_2 \succ Y_1$
10	−0.0998	−0.0764	−0.0252	−0.0478	$Y_3 \succ Y_4 \succ Y_2 \succ Y_1$
15	−0.0974	−0.0734	−0.0220	−0.0443	$Y_3 \succ Y_4 \succ Y_2 \succ Y_1$
20	−0.0959	−0.0714	−0.0198	−0.0420	$Y_3 \succ Y_4 \succ Y_2 \succ Y_1$

From Table 2, regardless of the value of the parameter δ , the priority of alternatives is always the same and the best option remains Y_3 , indicating that the parameter δ has little influence on the rankings of the alternatives. This proves the robustness of the proposed method from one aspect. To make the comprehensive evaluation more intuitive, the changes in the scoring results of each alternative manufacturer are shown in Figure 1.

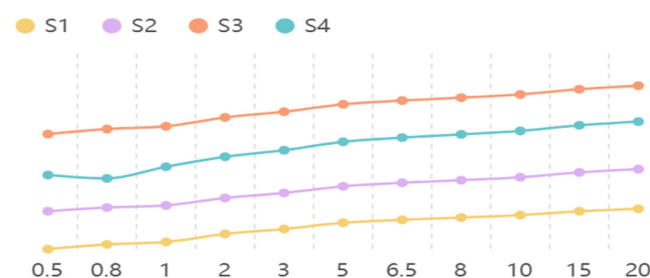


Figure 1. The influence of parameter δ on the scores of each alternative.

As can be seen from Figure 1, with the increase of parameter δ , the scoring of each scheme change accordingly and show an increasing trend as a whole, indicating that parameter has an obvious impact on the scoring results. However, the ranking of each scheme remains relatively stable, and it is always $Y_3 \succ Y_4 \succ Y_2 \succ Y_1$, showing that PIVFHFHCIG operator has certain stability.

The analysis results show that the aggregation results obtained by PIVFHFHCIG operator are stable and reasonable. Decision makers can choose different parameter values to sort the schemes by their own judgment or subjective attitude, which reflects certain flexibility. Meanwhile, they can also observe the dynamic change trend of the ranking by selecting a series of different parameter values. Thus, the PIVFHFHCIG operator is flexible and dynamic.

6.4. Comparative Analysis

In this section, a comparative analysis of the proposed Hamacher aggregation operators for PIVFHFHFs is presented to further demonstrate the rationality of the MADM method, including the PIVFHFHWA, PIVFHFHWG, and PIVFHFHCIA operators.

When the PIVFHFHCIA operator is used for information aggregation, the comprehensive scores of alternatives are given as (when $\delta = 0.8$):

$$\begin{aligned} M_1 &= \left\langle ([0.3155, 0.4680], [0.1030, 0.6703], 0.4615) \right\rangle \\ &\quad \left\langle ([0.3157, 0.3006], [0.0273, 0.4878], 0.5385) \right\rangle \\ M_2 &= \left\langle ([0.2722, 0.3740], [0.0526, 0.6237], 0.5000) \right\rangle \\ &\quad \left\langle ([0.2615, 0.2359], [0.0132, 0.5017], 0.5000) \right\rangle \\ M_3 &= \left\langle ([0.4585, 0.2957], [0.0259, 0.5063], 0.4177) \right\rangle \\ &\quad \left\langle ([0.3299, 0.4268], [0.0778, 0.5376], 0.5823) \right\rangle \\ M_4 &= \left\langle ([0.4213, 0.2686], [0.0190, 0.4785], 0.4250) \right\rangle \\ &\quad \left\langle ([0.3108, 0.3477], [0.0422, 0.6101], 0.5750) \right\rangle \end{aligned}$$

Then, the final scores of each alternative are: $S(Y_1) = -0.0674$, $S(Y_2) = -0.0425$, $S(Y_3) = 0.0245$, $S(Y_4) = -0.0097$. Thus, the manufacturers are ranked as $Y_3 \succ Y_4 \succ Y_2 \succ Y_1$.

Assume that the weighting vector given by experts is $\omega = (0.3, 0.25, 0.35, 0.1)^T$, then the scores using the PIVFHFHWA operator and PIVFHFHWG operator are (when $\delta = 0.8$):

$$S(Y_1) = -0.1337, S(Y_2) = -0.0561, S(Y_3) = 0.0629, S(Y_4) = -0.0148$$

$$S(Y_1) = -0.1921, S(Y_2) = -0.1249, S(Y_3) = -0.0246, S(Y_4) = -0.0748$$

Obviously, the ranking of enterprises is $Y_3 \succ Y_4 \succ Y_2 \succ Y_1$ when PIVFHFHWA and PIVFHFHWG operators are used to calculate the scores. Furthermore, this section analyzes and compares the scores and rankings of the four proposed operators at varying parameter values, as displayed in Table 3.

It can be seen from Table 3 that although the scores of each alternative obtained by PIVFHFHWA, PIVFHFHWG, PIVFHFHCIA, and PIVFHFHCIG operators are different, the rankings of alternatives are the same, that is, $Y_3 \succ Y_4 \succ Y_2 \succ Y_1$. Moreover, the aggregate values of PIVFHFHWA and PIVFHFHCIA operators are consistently higher than those of PIVFHFHWG and PIVFHFHCIG operators. The comprehensive evaluation values obtained by PIVFHFHWA and PIVFHFHCIA operators are decreasing with the increase of parameter, which means that the smaller the parameter value, the higher the optimism level of decision makers. However, the comprehensive scores of each alternative obtained by PIVFHFHWG and PIVFHFHCIG operators are increasing with the increase of parameter, which means that the smaller the parameter value, the higher the optimism level of decision makers. The reason may be that the Hamacher operator, as a fuzzy synthesis operator with parameters, has multiple monotone properties. The Hamacher t-norm is monotonically decreasing with

respect to parameter δ , while Hamacher t-conorm is monotonically increasing with respect to δ . Therefore, the former is more appropriate for optimistic decision makers and the latter for pessimistic decision makers. The Hamacher aggregation operators proposed in this paper are more widely used than the existing Fermatean fuzzy aggregation operators because they have considered the probability during the decision-making process and use interval numbers to represent the evaluation information, which can avoid information loss and reflect the interaction of each attribute.

Table 3. The scores and rankings obtained by different operators with respect to different δ .

	PIVFHFWG	PIVFHFWA	PIVFHFCIA	PIVFHFCIG
0.8	$S1 = -0.1921$ $S2 = -0.1249$ $S3 = -0.0246$ $S4 = -0.0748$	$S1 = -0.1337$ $S2 = -0.0501$ $S3 = 0.0629$ $S4 = -0.0148$	$S1 = -0.0704$ $S2 = -0.0425$ $S3 = 0.0245$ $S4 = -0.0097$	$S1 = -0.1179$ $S2 = -0.0951$ $S3 = -0.0466$ $S4 = -0.0717$
1	$S1 = -0.1904$ $S2 = -0.1234$ $S3 = -0.0223$ $S4 = -0.0732$	$S1 = -0.1355$ $S2 = -0.0581$ $S3 = 0.0599$ $S4 = -0.0158$	$S1 = -0.0714$ $S2 = -0.0436$ $S3 = 0.0224$ $S4 = -0.0107$	$S1 = -0.1164$ $S2 = -0.0938$ $S3 = -0.0449$ $S4 = -0.0699$
2	$S1 = -0.1848$ $S2 = -0.1182$ $S3 = -0.0154$ $S4 = -0.0678$	$S1 = -0.1409$ $S2 = -0.0639$ $S3 = 0.0504$ $S4 = -0.0189$	$S1 = -0.0855$ $S2 = -0.0466$ $S3 = 0.01604$ $S4 = -0.0136$	$S1 = -0.1114$ $S2 = -0.0892$ $S3 = -0.0394$ $S4 = -0.0637$
5	$S1 = -0.1772$ $S2 = -0.1104$ $S3 = -0.0076$ $S4 = -0.0598$	$S1 = -0.1465$ $S2 = -0.0706$ $S3 = 0.0383$ $S4 = -0.0227$	$S1 = -0.1077$ $S2 = -0.0500$ $S3 = 0.0086$ $S4 = -0.0172$	$S1 = -0.1045$ $S2 = -0.0820$ $S3 = -0.0313$ $S4 = -0.0545$
8	$S1 = -0.1736$ $S2 = -0.1063$ $S3 = -0.0043$ $S4 = -0.0557$	$S1 = -0.1488$ $S2 = -0.0737$ $S3 = 0.0326$ $S4 = -0.0246$	$S1 = -0.1204$ $S2 = -0.0514$ $S3 = 0.0054$ $S4 = -0.0188$	$S1 = -0.1013$ $S2 = -0.0782$ $S3 = -0.0272$ $S4 = -0.0499$
15	$S1 = -0.1696$ $S2 = -0.1014$ $S3 = -0.0007$ $S4 = -0.0506$	$S1 = -0.1514$ $S2 = -0.0772$ $S3 = 0.0260$ $S4 = -0.0269$	$S1 = -0.1381$ $S2 = -0.0530$ $S3 = 0.0017$ $S4 = -0.0209$	$S1 = -0.0974$ $S2 = -0.0734$ $S3 = -0.0220$ $S4 = -0.0443$

7. Conclusions

This paper discusses several new Hamacher aggregation operators for PIVFHFSSs and relevant applications. Firstly, this paper introduces probability into the interval-valued Fermatean hesitant fuzzy environment and develops the concept of PIVFHFSSs. Based on the Hamacher operation, two probabilistic interval-valued Fermatean hesitant fuzzy Hamacher aggregation operators are defined: the PIVFHFHWA operator and PIVFHFHWG operator, and their characteristics and special forms are described. Then, the PIVFHFHCIA and PIVFHFHCIG operators are proposed based on the Choquet integral. These new operators have some excellent properties, such as boundedness, permutation invariance, and monotonicity. They consider the interaction between different attributes and the personal preferences of decision makers, effectively improving the rationality of decision results and avoiding information loss. Finally, a method with PIVFHFSSs information for solving MADM problems is developed and illustrated through a numerical example of reducing carbon emissions of manufacturers in the supply chain. The stability and feasibility of this method are verified by sensitivity analysis and comparative analysis. The presented MADM method not only expands the Hamacher operator and its applications but also combines probabilistic interval-valued information with FHFSS to enrich the theory and practice of FFS.

However, the fuzzy measure in this paper is directly given by experts according to the practical significance of each attribute and decision requirement, which may contain some subjective evaluations. Meanwhile, the collected evaluation information in the decision matrix of this paper is not fully combined with actual enterprises, so it is necessary to improve the practical application of the presented method and explore some new data collection methods under the background of big data. Therefore, evaluation attributes' importance degree and weighting information should be further studied. In fuzzy decision making, it is a new attempt to combine probability interval-valued information with FHFSS. A new concept of PIVFHFSSs is proposed, and some Hamacher aggregation operators of

PIVFHFSs are developed in this paper. In future research, it is necessary to explore more studies and construct a relatively complete theoretical system of PIVFHFSs, including new aggregation operators, distance measures, correlation coefficients, entropy measures, and so on. Then, PIVFHFSs can be combined with some classical decision-making methods and further applied to more fields to improve their applicability, such as cluster analysis, pattern recognition, and medical diagnosis. Furthermore, we can combine the probabilistic interval-valued information with other fuzzy sets to develop new concepts, such as neutrosophic sets [57], fuzzy rough sets [58], complex fuzzy sets [59], and dual hesitant sets [60].

Author Contributions: Conceptualization, C.R.; methodology, C.R. and X.C.; validation, C.R. and X.C.; formal analysis, C.R.; data curation, C.R. and X.C.; writing—original draft preparation, C.R.; writing—review and editing, C.R. and X.C.; visualization, C.R.; supervision, C.R.; funding acquisition, C.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Guangdong Provincial Philosophy and Social Science Planning Project, grant number GD23XGL012 and the Innovative Team Project of Guangdong Universities, grant number 2019WCXTD008.

Data Availability Statement: Our data are derived from original calculations, so we have no data to offer.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

FSs	fuzzy sets
IFS	intuitionistic fuzzy set
HFSs	hesitant fuzzy sets
PFSs	Pythagorean fuzzy sets
q-ROFS	q-rung orthopair fuzzy set
FFSs	Fermatean fuzzy sets
FFLTs	Fermatean fuzzy linguistic term sets
FFSSs	Fermatean fuzzy soft sets
ELECTRE	ELimination Et Choix Traduisant la REalité
MCGDM	multi-criteria group decision-making
FHFS	Fermatean hesitant fuzzy set
VIKOR	vlsekriterijumska optimizacija i kompromisno resenje
IVIFS	interval-valued intuitionistic fuzzy set
IVPFS	interval-valued Pythagorean fuzzy set
IVHFS	interval-valued hesitant fuzzy set
IVPHFS	interval-valued Pythagorean hesitant fuzzy set
IVFFSs	interval-valued Fermatean fuzzy sets
IVFHFSs	interval-valued Fermatean hesitant fuzzy sets
MADM	multi-attribute decision-making
PHFSs	probabilistic hesitant fuzzy sets
PHFEs	probabilistic hesitant fuzzy elements
PHFPR	probabilistic hesitant fuzzy preference relation
PIVIHFSs	probabilistic interval-valued intuitionistic hesitant fuzzy sets
PIVHFS	probabilistic interval-valued hesitant fuzzy set
TOPSIS	Technique for Order Preference by Similarity to an Ideal Solution
BM	Bonferroni mean operator
OWA	ordered weighted averaging operator
I-IVIFHOWG	induced interval-valued intuitionistic fuzzy Hamacher ordered weighted geometric
IVDHFSs	interval-valued dual hesitant fuzzy sets

IVPFCIG	interval-valued Pythagorean fuzzy Choquet integral geometric
PIVHPFHCIG	probabilistic interval-valued hesitant Pythagorean fuzzy Hamacher Choquet integral geometric operator
PIVFHFSs	probabilistic interval-valued Fermatean hesitant fuzzy sets
PIVFHFHWA	probabilistic interval-valued Fermatean hesitant fuzzy Hamacher weighted averaging operator
PIVFHFHWG	probabilistic interval-valued Fermatean fuzzy Hamacher weighted geometric operator
PIVFHFHCIA	probabilistic interval-valued Fermatean hesitant fuzzy Hamacher Choquet integral averaging operator
PIVFHFHCIG	probabilistic interval-valued Fermatean hesitant fuzzy Hamacher Choquet integral geometric operator
IVFFN	interval-valued Fermatean fuzzy number
IVFHFN	interval-valued Fermatean hesitant fuzzy number
CoCoSo	combined compromise solution
PIVFHFEs	probabilistic interval-valued Fermatean hesitant fuzzy elements

References

- Zadeh, L.A. Fuzzy sets. *Inf. Control.* **1965**, *8*, 338–353. [\[CrossRef\]](#)
- Atanassov, K.T. Intuitionistic fuzzy sets. *Fuzzy Sets. Syst.* **1986**, *20*, 87–96. [\[CrossRef\]](#)
- Atanassov, K.; Sotirova, E.; Andonov, V. Generalized Net Model of Multicriteria Decision Making Procedure Using Intercriteria Analysis. In *Advances in Fuzzy Logic and Technology 2017*; Kacprzyk, J., Szmidt, E., Zadrozny, S., Atanassov, K., Krawczak, M., Eds.; Springer: Cham, Switzerland, 2017; Volume 641.
- Zhang, X.; Zhou, B.; Li, P. A general frame for intuitionistic fuzzy rough sets. *Inf. Sci.* **2012**, *216*, 34–49. [\[CrossRef\]](#)
- Mahmood, T.; Ali, W.; Ali, Z.; Chinram, R. Power aggregation operators and similarity measures based on improved intuitionistic hesitant fuzzy sets and their applications to multiple attribute decision making. *Comp. Model. Eng. Sci.* **2021**, *126*, 1165–1187. [\[CrossRef\]](#)
- Yuan, R.; Tang, J.; Meng, F. Linguistic intuitionistic fuzzy group decision making based on aggregation operators. *Int. J. Fuzzy Syst.* **2019**, *21*, 407–420. [\[CrossRef\]](#)
- Torra, V. Hesitant fuzzy sets. *Int. J. Intell. Syst.* **2010**, *25*, 529–539. [\[CrossRef\]](#)
- Yager, R.R. Pythagorean membership grades in multicriteria decision making. *IEEE Trans. Fuzzy Syst.* **2013**, *22*, 958–965. [\[CrossRef\]](#)
- Yager, R.R. Generalized orthopair fuzzy sets. *IEEE Trans. Fuzzy Syst.* **2016**, *25*, 1222–1230. [\[CrossRef\]](#)
- Senapati, T.; Yager, R.R. Fermatean fuzzy sets. *J. Amb. Intell. Hum. Comput.* **2020**, *11*, 663–674. [\[CrossRef\]](#)
- Senapati, T.; Yager, R.R. Some new operations over Fermatean fuzzy numbers and application of Fermatean fuzzy WPM in multiple criteria decision making. *Informatica* **2019**, *30*, 391–412. [\[CrossRef\]](#)
- Aydemir, S.B.; Yilmaz Gunduz, S. Fermatean fuzzy TOPSIS method with Dombi aggregation operators and its application in multi-criteria decision making. *J. Intell. Fuzzy Syst.* **2020**, *39*, 851–869. [\[CrossRef\]](#)
- Liu, D.; Liu, Y.; Chen, X. Fermatean fuzzy linguistic set and its application in multicriteria decision making. *Int. J. Intell. Syst.* **2019**, *34*, 878–894. [\[CrossRef\]](#)
- Hadi, A.; Khan, W.; Khan, A. A novel approach to MADM problems using Fermatean fuzzy Hamacher aggregation operators. *Int. J. Intell. Syst.* **2021**, *36*, 3464–3499. [\[CrossRef\]](#)
- Deng, Z.; Wang, J. Evidential Fermatean fuzzy multicriteria decision-making based on Fermatean fuzzy entropy. *Int. J. Intell. Syst.* **2021**, *36*, 5866–5886. [\[CrossRef\]](#)
- Zeb, A.; Khan, A.; Juniad, M.; Izhar, M. Fermatean fuzzy soft aggregation operators and their application in symptomatic treatment of COVID-19 (case study of patients identification). *J. Amb. Intell. Hum. Comput.* **2022**, *14*, 11607–11624. [\[CrossRef\]](#)
- Ganie, A.H. Multicriteria decision-making based on distance measures and knowledge measures of Fermatean fuzzy sets. *Granular Comput.* **2022**, *7*, 979–998. [\[CrossRef\]](#)
- Kirişci, M.; Demir, I.; Şimşek, N. Fermatean fuzzy ELECTRE multi-criteria group decision-making and most suitable biomedical material selection. *Artif. Intell. Med.* **2022**, *127*, 102278. [\[CrossRef\]](#)
- Ruan, C.Y.; Chen, X.J.; Han, L.N. Fermatean Hesitant Fuzzy Prioritized Heronian Mean Operator and Its Application in Multi-Attribute Decision Making. *Comput. Mater. Con.* **2023**, *75*, 3203–3222. [\[CrossRef\]](#)
- Mishra, A.R.; Chen, S.M.; Rani, P. Multiattribute decision making based on Fermatean hesitant fuzzy sets and modified VIKOR method. *Inf. Sci.* **2022**, *607*, 1532–1549. [\[CrossRef\]](#)
- Wang, Y.; Ma, X.; Qin, H.; Sun, H.; Wei, W. Hesitant Fermatean fuzzy Bonferroni mean operators for multi-attribute decision-making. *Complex Intell. Syst.* **2023**, *1*–33. [\[CrossRef\]](#)
- Atanassov, K.T.; Gargov, G. Interval-valued intuitionistic fuzzy sets. *Fuzzy Sets. Syst.* **1989**, *3*, 343–349. [\[CrossRef\]](#)
- Atanassov, K.T. Operators over interval-valued intuitionistic fuzzy sets. *Fuzzy Sets. Syst.* **1994**, *64*, 159–174. [\[CrossRef\]](#)

24. Nayagam, V.L.G.; Sivaraman, G. Ranking of interval-valued intuitionistic fuzzy sets. *Appl. Soft Comput.* **2011**, *11*, 3368–3372. [\[CrossRef\]](#)
25. Chen, N.; Xu, Z.; Xia, M. Interval-valued hesitant preference relations and their applications to group decision making. *Knowl. Based. Syst.* **2013**, *37*, 528–540. [\[CrossRef\]](#)
26. Peng, X.; Yang, Y. Fundamental properties of interval-valued Pythagorean fuzzy aggregation operators. *Int. J. Intell. Syst.* **2016**, *31*, 444–487. [\[CrossRef\]](#)
27. Zhang, M.; Zheng, T.; Zheng, W.; Zhou, L. Interval-valued pythagorean hesitant fuzzy set and its application to multiattribute group decision-making. *Complexity* **2020**, *2020*, 1–26. [\[CrossRef\]](#)
28. Jeevaraj, S. Ordering of interval-valued Fermatean fuzzy sets and its applications. *Expert Syst. Appl.* **2021**, *185*, 115613.
29. Akram, M.; Shah, S.M.U.; Al-Shamiri, M.M.A.; Edalatpanah, S.A. Fractional transportation problem under interval-valued Fermatean fuzzy sets. *AIMS Math.* **2022**, *7*, 17327–17348. [\[CrossRef\]](#)
30. Mishra, A.R.; Liu, P.; Rani, P. COPRAS method based on interval-valued hesitant Fermatean fuzzy sets and its application in selecting desalination technology. *Appl. Soft Comput.* **2022**, *119*, 108570. [\[CrossRef\]](#)
31. Qin, H.; Peng, Q.; Ma, X.; Zhan, J. A new multi-attribute decision making approach based on new score function and hybrid weighted score measure in interval-valued Fermatean fuzzy environment. *Complex Intell. Syst.* **2023**, *9*, 5359–5376. [\[CrossRef\]](#)
32. Sergi, D.; Sari, I.U.; Senapati, T. Extension of capital budgeting techniques using interval-valued Fermatean fuzzy sets. *J. Intell. Fuzzy Syst.* **2022**, *42*, 365–376. [\[CrossRef\]](#)
33. Rani, P.; Mishra, A.R. Interval-valued fermatean fuzzy sets with multi-criteria weighted aggregated sum product assessment-based decision analysis framework. *Neural Comput. Appl.* **2022**, *34*, 8051–8067. [\[CrossRef\]](#) [\[PubMed\]](#)
34. Demir, I. Novel correlation coefficients for interval-valued Fermatean hesitant fuzzy sets with pattern recognition application. *Turkish J. Math.* **2023**, *47*, 213–233. [\[CrossRef\]](#)
35. Zhang, S.; Xu, Z.; He, Y. Operations and integrations of probabilistic hesitant fuzzy information in decision making. *Inform. Fusion* **2017**, *38*, 1–11. [\[CrossRef\]](#)
36. Jiang, F.; Ma, Q. Multi-attribute group decision making under probabilistic hesitant fuzzy environment with application to evaluate the transformation efficiency. *Appl. Intell.* **2018**, *48*, 953–965. [\[CrossRef\]](#)
37. Li, J.; Wang, Z.X. Multi-attribute decision making based on prioritized operators under probabilistic hesitant fuzzy environments. *Soft Comput.* **2019**, *23*, 3853–3868. [\[CrossRef\]](#)
38. De, A.; Das, S.; Kar, S. Multiple attribute decision making based on probabilistic interval-valued intuitionistic hesitant fuzzy set and extended TOPSIS method. *J. Intell. Fuzzy Syst.* **2019**, *37*, 5229–5248. [\[CrossRef\]](#)
39. Garg, H. Some methods for strategic decision-making problems with immediate probabilities in Pythagorean fuzzy environment. *Int. J. Intell. Syst.* **2018**, *33*, 687–712. [\[CrossRef\]](#)
40. Ali, J.; Bashir, Z.; Rashid, T. On distance measure and TOPSIS model for probabilistic interval-valued hesitant fuzzy sets: Application to healthcare facilities in public hospitals. *Grey Syst.* **2022**, *12*, 197–229. [\[CrossRef\]](#)
41. Bonferroni, C. Sulle medie multiple di potenze. *Boll. Dell'unione Mat. Ital.* **1950**, *5*, 267–270.
42. Rani, P.; Mishra, A.R. Fermatean fuzzy Einstein aggregation operators-based MULTIMOORA method for electric vehicle charging station selection. *Expert Syst. Appl.* **2021**, *182*, 115267. [\[CrossRef\]](#)
43. Yager, R.R. On ordered weighted averaging aggregation operators in multicriteria decision making. *IEEE Trans. Syst. Man. Cybern.* **1988**, *18*, 183–190. [\[CrossRef\]](#)
44. Li, L.G.; Peng, D.H. Interval-valued hesitant fuzzy Hamacher synergetic weighted aggregation operators and their application to shale gas areas selection. *Math. Prob. Eng.* **2014**, *2014*, 1–25. [\[CrossRef\]](#)
45. Liu, P. Some Hamacher aggregation operators based on the interval-valued intuitionistic fuzzy numbers and their application to group decision making. *IEEE Trans. Fuzzy Syst.* **2013**, *22*, 83–97. [\[CrossRef\]](#)
46. Xiao, S. Induced interval-valued intuitionistic fuzzy Hamacher ordered weighted geometric operator and their application to multiple attribute decision making. *J. Intell. Fuzzy Syst.* **2014**, *27*, 527–534. [\[CrossRef\]](#)
47. Senapati, T.; Chen, G. Some novel interval-valued Pythagorean fuzzy aggregation operator based on Hamacher triangular norms and their application in MADM issues. *Comput. Appl. Math.* **2021**, *40*, 109. [\[CrossRef\]](#)
48. Shahzadi, G.; Luqman, A.; Karaaslan, F. A decision-making technique under interval-valued Fermatean fuzzy Hamacher interactive aggregation operators. *Soft Comput.* **2023**, 1–28. [\[CrossRef\]](#)
49. Murofushi, T.; Sugeno, M. An interpretation of fuzzy measures and the Choquet integral as an integral with respect to a fuzzy measure. *Fuzzy Sets. Syst.* **1989**, *29*, 201–227. [\[CrossRef\]](#)
50. Choquet, G. Theory of capacities. *Proc. Ann. De L'institut Fourier* **1954**, *5*, 131–295.
51. Zhang, S.; Yu, D. Some geometric Choquet aggregation operators using Einstein operations under intuitionistic fuzzy environment. *J. Intell. Fuzzy Syst.* **2014**, *26*, 491–500. [\[CrossRef\]](#)
52. Khan, M.S.A.; Abdullah, S.; Ali, A.; Amin, F.; Hussain, F. Pythagorean hesitant fuzzy Choquet integral aggregation operators and their application to multi-attribute decision-making. *Soft Comput.* **2019**, *23*, 251–267. [\[CrossRef\]](#)
53. Qu, G.; Qu, W.; Li, C. Some new interval-valued dual hesitant fuzzy Choquet integral aggregation operators and their applications. *J. Intell. Fuzzy Syst.* **2018**, *34*, 245–266. [\[CrossRef\]](#)
54. Sajjad Ali Khan, M.; Abdullah, S.; Yousaf Ali, M.; Hussain, I.; Farooq, M. Extension of TOPSIS method base on Choquet integral under interval-valued Pythagorean fuzzy environment. *J. Intell. Fuzzy Syst.* **2018**, *34*, 267–282. [\[CrossRef\]](#)

55. Luo, S.; Liu, J. The probabilistic interval-valued hesitant pythagorean fuzzy set and its application in selecting processes of project private partner. *IEEE Access* **2019**, *7*, 170304–170321. [[CrossRef](#)]
56. Shao, S.; Zhang, X.; Zhao, Q. Multi-attribute decision making based on probabilistic neutrosophic hesitant fuzzy choquet aggregation operators. *Symmetry* **2019**, *11*, 623. [[CrossRef](#)]
57. Abdel-Basset, M.; Mohamed, M.; Mostafa, N.N.; El-Henawy, I.M.; Abouhawwash, M. New multi-criteria decision-making technique based on neutrosophic axiomatic design. *Sci. Rep.* **2022**, *12*, 10657. [[CrossRef](#)]
58. Zhan, J.; Zhang, X.; Yao, Y. Covering based multigranulation fuzzy rough sets and corresponding applications. *Artif. Intell. Rev.* **2020**, *53*, 1093–1126. [[CrossRef](#)]
59. Sobhi, S.; Dick, S. An investigation of complex fuzzy sets for large-scale learning. *Fuzzy Sets. Syst.* **2023**, *471*, 108660. [[CrossRef](#)]
60. Ali, G.; Afzal, A.; Sheikh, U.; Nabeel, M. Multi-criteria group decision-making based on the combination of dual hesitant fuzzy sets with soft expert sets for the prediction of a local election scenario. *Granular Comput.* **2023**, 1–28. [[CrossRef](#)]

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