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Processing Fractional Differential Equations Using ψ-Caputo Derivative

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Abstract: Recently, many scientists have studied a wide range of strategies for solving characteristic types of symmetric differential equations, including symmetric fractional differential equations (FDEs). In our manuscript, we obtained sufficient conditions to prove the existence and uniqueness of solutions (EUS) for FDEs in the sense ψ-Caputo fractional derivative (ψ-CFD) in the second-order $1 < \alpha < 2$. We know that ψ-CFD is a generalization of previously familiar fractional derivatives: Riemann-Liouville and Caputo. By applying the Banach fixed-point theorem (BFPT) and the Schauder fixed-point theorem (SFPT), we obtained the desired results, and to embody the theoretical results obtained, we provided two examples that illustrate the theoretical proofs.

Keywords: ψ-Caputo fractional derivative; existence and uniqueness; fixed-point theorems

MSC: 34B10; 34B15; 26A33

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1. Introduction

Fractional calculus (FC) is a branch of calculus that deals with integrals and derivatives of non-integer orders. The history of FC can be traced back to the 17th century, when the German mathematician Gottfried Leibniz first mentioned the concept of fractional differentiation in a letter to his colleague Johann Bernoulli. However, the development of FCs as a field of study actually began in the 19th century, with the work of several mathematicians, including Augustin-Louis Cauchy, Liouville, and Riemann. In the early 20th century, the French mathematician Paul Lévy used fractional calculus to model random processes, and it was subsequently used in the study of fractals and other areas of mathematics.

Symmetrical FDEs have applications in various areas of science and engineering, including physics, signal processing, and control theory. They are particularly useful for modeling symmetric physical phenomena, such as waves, vibrations, and oscillations. Solving symmetrical fractional differential equations can be challenging, and various numerical and analytical techniques have been developed to tackle this problem (see [1,2]). Symmetrical FC has many applications in physics, engineering, finance, and other fields. For example, FDEs have been used to model anomalous diffusion in materials, and fractional control systems have been developed for applications in robotics and automation. FC has also been used in finance to model the behaviors of stock prices and other financial variables (see [3–6] and the references cited therein).

Overall, the study of FC has proven to be a valuable tool for understanding a wide range of phenomena in the natural world, and has led to many important applications in science and engineering. The EUS to FDEs is an important topic in the field of FC. In general, FDEs can have a more complex solution behavior than their integer-order counterparts. However, under certain conditions, it is possible to establish the EUS to these equations. One approach to studying the EUS to FDEs is to use fixed-point theorems such as the BFPT or the SFPT. These theorems provide conditions under which a unique solution exists for a given FDE. Applications of FDEs are numerous and include the modeling of anomalous diffusion, viscoelastic materials, and biological systems. In particular, FDEs have been used to model the behaviors of cells and tissues, as well as the spread of diseases and other biological phenomena (see [7–9]). FDEs have also found applications in the modeling of financial markets and the analysis of stock price dynamics. For example, the famous Black-Scholes model for option pricing can be generalized using FC to account for non-Gaussian price fluctuations. Overall, the EUS to FDEs is an important topic with many applications in science and engineering. Further research in this area is likely to lead to new insights and advances in the modeling and analysis of complex systems. The fixed-point theorem is a fundamental result in mathematics that provides conditions under which a function has a fixed point, that is, a point that remains unchanged under the function. The history of fixed-point theorems can be traced back to the early 19th century, when the French mathematician Augustin-Louis Cauchy first introduced the concept of a fixed point in his work on iterative methods (see [10–12]).

The first general fixed-point theorem was proved by the German mathematician Carl Friedrich Gauss in the 19th century, and since then, many other fixed-point theorems have been established in various areas of mathematics. The importance of fixed-point theorems for FC lies in their application to the study of FDEs. Many FDEs can be written in the form of an integral equation, and fixed-point theorems provide conditions under which a unique solution to these equations exists. In particular, the BFPT and the SFPT have been used extensively in the study of FDEs. These theorems provide sufficient conditions for the EUS to integral equations, which can be used to model a wide range of phenomena in science and engineering. Overall, fixed-point theorems play a crucial role in the study of FC and have many important applications in fields such as physics, engineering, and finance. By providing a powerful tool for the analysis of complex systems, fixed-point theorems have enabled researchers to make significant advances in our understanding of the natural world.

The theory of the existence of solutions to fractional differential models that are from acquired investigations has drawn the attention of many authors (see [13–19]). Most of them recognized the use of the derivatives of Riemann-Liouville and Caputo derivatives in representing the basic FDE (see [20–23]). There is also another type of fractional derivative, the ψ-Caputo derivative, which is a generalization of the previously mentioned derivatives and was introduced in [24,25]. This derivative differs from other derivatives in the sense that the kernel of integration is the generalization of the Hadamard derivative.

It is known that there are many types of fractional derivatives that can be used to determine the EUS of FDEs. Recently, many authors have focused on the ψ-Caputo fractional derivative, and our work is a continuation in this stream. The field of fractional calculus has shown and supported the sense of Caputo due its accuracy in modeling different phenomenal effects; this stems from the fact that the initial data for dynamical systems are like those of classical ones. However, other types of derivatives such as the Atangana-Baleanu derivative ([26]) will be considered in a forthcoming work.

Detailed concepts of ψ-CFD and integral can be found in ([9,27–30]). The new addition to our research is the idea of the ψ-Caputo fractional operator.

In a clearer concept, we consider the nonlinear ψ-Caputo given by

$$\begin{cases} \left(\mathcal{D}_a^{\alpha;\psi} + \gamma\mathcal{D}_a^{\alpha-1;\psi}\right)\varphi(\varsigma) = f(\varsigma, \varphi(\varsigma)), \ \varsigma \in [a, \Im], \\ \varphi(a) = \varphi'(a) = 0, \end{cases} \tag{1}$$

where $1 < \alpha < 2$, $\varsigma \in [a, \Im]$, $1 \leq a < \Im$, $f : J \times \mathbb{R} \to \mathbb{R}$ are given continuous functions (CFs), and γ is a positive real number. Let $\psi : [a, \Im] \to \mathbb{R}$ be increasing via ψ' bounded and $\psi'(\varsigma) \neq 0$, for all ς. The symbol $\mathcal{C}(J, \mathbb{R})$ represents the Banach space of CFs φ from J to $\mathbb{R}$ with the norm $\|\varphi\| = \sup\{|\varphi(\varsigma)| : \varsigma \in J\}$.

2. Essential Preliminaries

In this session, we provide definitions and features of the previous familiar derivatives and the ψ-fractional derivative (see [9,27,28]).

Definition 1 ([9])**.** *Let $\varphi : (0, \infty) \to \mathbb{R}$ be a CF. Then, the Riemann-Liouville fractional derivative (RLFD) of order $\alpha > 0, n = [\alpha] + 1$, ($[\alpha]$ indicates the integer part of the real number α), defined as*

$$^{RL}D^{\alpha}_{0+}\varphi(\varsigma) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{d\varsigma})^n \int_0^{\varsigma} (\varsigma - \tau)^{n-\alpha-1}\varphi(\tau)d\tau,$$

where $n - 1 < \alpha < n$.

Definition 2 ([9])**.** *Let $\varphi : (0, \infty) \to \mathbb{R}$ be a CF. Then, the CFD of order $\alpha > 0, n = [\alpha] + 1$, is defined as*

$$^{C}D^{\alpha}_{0+}\varphi(\varsigma) = \frac{1}{\Gamma(n - \alpha)} \int_0^{\varsigma} (\varsigma - \tau)^{n-\alpha-1}\varphi^{(n)}(\tau)d\tau,$$

where $n - 1 < \alpha < n$.

Definition 3 ([9])**.** *The Hadamard fractional integral (HFI) of order $\alpha > 0$ for a CF φ:$[1, +\infty) \to \mathbb{R}$ is defined as*

$$^{H}\mathfrak{I}^{\alpha}_{1}\varphi(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \varphi(\tau)\frac{d\tau}{\tau}.$$

Definition 4 ([9])**.** *The CHD of order $\alpha > 0$ for a CF $\varphi : [1, +\infty) \to \mathbb{R}$ is defined as*

$$^{CH}\mathfrak{D}^{\alpha}_{1}\varphi(t) = \frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\log \frac{t}{\tau}\right)^{n-\alpha-1} \delta^n\varphi(\tau)\frac{d\tau}{\tau},\ n - 1 < \alpha < n,$$

where $\delta^n = \left(t\frac{d}{dt}\right)^n, n \in \mathbb{N}$.

Definition 5 ([9,27])**.** *The ψ-Riemann-Liouville fractional integral (ψ-RLFI) of order $\alpha > 0$ for a CF $\varphi : [a, \Im] \to \mathbb{R}$ is referred to as*

$$\mathcal{I}^{\alpha;\psi}_{a}\varphi(\varsigma) = \int_a^{\varsigma} \frac{(\psi(\varsigma) - \psi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\psi'(\tau)\varphi(\tau)d\tau.$$

Definition 6 ([28])**.** *The CFD of order $\alpha > 0$ for a $\varphi : [0, +\infty) \to \mathbb{R}$ is intended by*

$$D^{\alpha}\varphi(\varsigma) = \frac{1}{\Gamma(n-\alpha)} \int_0^{\varsigma} (\varsigma - \tau)^{n-\alpha-1}\varphi^{(n)}(\tau)d\tau,\ n - 1 < \alpha < n.$$

Definition 7 ([9,27])**.** *The ψ-Caputo fractional derivative (ψ-CFD) of order $\alpha > 0$ for a CF $\varphi : [a, \Im] \to \mathbb{R}$ is the aim of*

$$\mathcal{D}^{\alpha;\psi}_{a}\varphi(\varsigma) = \int_a^{\varsigma} \frac{(\psi(\varsigma) - \psi(\tau))^{n-\alpha-1}}{\Gamma(n-\alpha)}\psi'(\tau)\partial^n_{\psi}\varphi(\tau)d\tau,\ \varsigma > a,\ n - 1 < \alpha < n,$$

where $\partial^n_{\psi} = \left(\frac{1}{\psi'(\varsigma)}\frac{d}{d\varsigma}\right)^n, n \in \mathbb{N}$.

Lemma 1 ([9,27])**.** *Let $q, \ell > 0$, and $\varphi \in \mathcal{C}([a\ ,\ b],\ \mathbb{R})$. Then, $\forall \varsigma \in [a\ ,\ b]$, and by assuming $F_a(\varsigma) = \psi(\varsigma) - \psi(a)$, we have*

1. $\mathcal{I}_a^{q;\psi}\mathcal{I}_a^{\ell;\psi}\varphi(\varsigma) = \mathcal{I}_a^{q+\ell;\psi}\varphi(\varsigma)$,
2. $\mathcal{D}_a^{q;\psi}\mathcal{I}_a^{q;\psi}\varphi(\varsigma) = \varphi(\varsigma)$,
3. $\mathcal{I}_a^{q;\psi}(F_a(\varsigma))^{\ell-1} = \dfrac{\Gamma(\ell)}{\Gamma(\ell+q)}(F_a(\varsigma))^{\ell+q-1}$,
4. $\mathcal{D}_a^{q;\psi}(F_a(\varsigma))^{\ell-1} = \dfrac{\Gamma(\ell)}{\Gamma(\ell-q)}(F_a(\varsigma))^{\ell-q-1}$,
5. $\mathcal{D}_a^{q;\psi}(F_a(\varsigma))^{k} = 0$, *for* $k \in \{0,\ \ldots\ ,\ n-1\}$, $n \in \mathbb{N}$, $q \in (n-1, n]$.

Lemma 2 ([9,27])**.** *Let $n-1 < \alpha_1 \le n, \alpha_2 > 0,\ a > 0,\ \varphi \in L(a, \mathcal{T}),\ \mathcal{D}_a^{\alpha_1;\psi}\varphi \in L(a, \mathcal{T})$. Then, the differential equation*

$$\mathcal{D}_a^{\alpha_1;\psi}\varphi = 0$$

has the unique solution

$$\varphi(\varsigma) = \mathcal{W}_0 + \mathcal{W}_1(\psi(\varsigma) - \psi(a)) + \mathcal{W}_2(\psi(\varsigma) - \psi(a))^2 + \cdots + \mathcal{W}_{n-1}(\psi(\varsigma)\ -\ \psi(a))^{n-1},$$

and

$$\begin{aligned}\mathcal{I}_a^{\alpha_1;\psi}\mathcal{D}_a^{\alpha_1;\psi}\varphi(\varsigma) = \varphi(\varsigma) + \mathcal{W}_0 + \mathcal{W}_1(\psi(\varsigma) - \psi(a)) + \mathcal{W}_2(\psi(\varsigma) - \psi(a))^2 \\ + \cdots + \mathcal{W}_{n-1}(\psi(\varsigma) - \psi(a))^{n-1},\end{aligned}$$

with $\mathcal{W}_\ell \in \mathbb{R}$, $\ell = 0, 1,\ \ldots\ , n-1$.

Furthermore,

$$\mathcal{D}_a^{\alpha_1;\psi}\mathcal{I}_a^{\alpha_1;\psi}\varphi(\varsigma) = \varphi(\varsigma),$$

and

$$\mathcal{I}_a^{\alpha_1;\psi}\mathcal{I}_a^{\alpha_2;\psi}\varphi(\varsigma) = \mathcal{I}_a^{\alpha_2;\psi}\mathcal{I}_a^{\alpha_1;\psi}\varphi(\varsigma) = \mathcal{I}_a^{\alpha_1+\alpha_2;\psi}\varphi(\varsigma).$$

Theorem 1. *Let $f \in \mathcal{C}(J, \mathbb{R})$, and $\varphi \in \mathcal{C}_\delta^2(J, \mathbb{R})$. The fractional linear differential equation*

$$\begin{aligned}&\left(\mathcal{D}_a^{\alpha;\psi} + \gamma\mathcal{D}_a^{\alpha-1;\psi}\right)\varphi(\varsigma) = f(\varsigma),\ 1 < \alpha < 2, \\ &\varphi(a) = \varphi'(a) = 0,\end{aligned} \tag{2}$$

has a solution given by

$$\varphi(\varsigma) = (\alpha-1)e^{-\gamma\psi(\varsigma)}\int_a^{\varsigma}\psi'(\tau)e^{-\gamma\psi(\tau)}\mathcal{I}_a^{\alpha-1;\psi}f(\tau)d\tau. \tag{3}$$

Proof. Taking the ψ-fractional integral $\mathcal{I}_a^{\alpha;\psi}$ to both sides of Equation (2), we obtain

$$\mathcal{I}_a^{\alpha;\psi}\left(\mathcal{D}_a^{\alpha;\psi}(\varphi(\varsigma))\right) + \gamma\mathcal{I}_a^{1}\left(\mathcal{I}_a^{\alpha-1;\psi}\mathcal{D}_a^{\alpha-1;\psi}(\varphi(\varsigma))\right) = \mathcal{I}_a^{\alpha;\psi}(f(\varsigma)).$$

Using Lemma 2, this implies that

$$\begin{aligned}&\left(\varphi(\varsigma) - b_1(\psi(\varsigma) - \psi(a))^{\alpha-1} - b_2(\psi(\varsigma) - \psi(a))^{\alpha-2}\right) \\ &+ \gamma\mathcal{I}_a^{1}\left(\varphi(\varsigma) - d_1(\psi(\varsigma) - \psi(a))^{\alpha-2}\right) = \mathcal{I}_a^{\alpha;\psi}(f(\varsigma)),\end{aligned}$$

which implies that

$$\begin{aligned}&\varphi(\varsigma) - b_1(\psi(\varsigma) - \psi(a))^{\alpha-1} - b_2(\psi(\varsigma) - \psi(a))^{\alpha-2} \\ &+ \frac{\gamma}{\Gamma(1)}\int_a^{\varsigma}\left(\varphi(\tau) - d_1(\psi(\tau) - \psi(a))^{\alpha-2}\right)\psi'(\tau)d\tau = \mathcal{I}_a^{\alpha;\psi}(f(\varsigma)).\end{aligned} \tag{4}$$

The initial condition $\varphi(a) = 0$, leads to $b_2 = 0$. Obviously, we can take the first ordinary derivative of the Equation (4),

$$\varphi'(\varsigma) + \gamma\psi'(\varsigma)\varphi(\varsigma) = (b_1(\alpha-1) + \gamma d_1)\psi'(\varsigma)(\psi(\varsigma) - \psi(a))^{\alpha-2} + (\alpha-1)\psi'(\varsigma)\mathcal{I}_a^{\alpha-1;\psi} f(\varsigma).$$

The condition $\varphi'(a) = 0$, implies that $b_1(\alpha-1) + \gamma d_1 = 0$.

Let $\varphi(\varsigma) = e^{-\gamma\psi(\varsigma)}u(\varsigma)$, $\varphi'(\varsigma) = -\gamma\psi'(\varsigma)e^{-\gamma\psi(\varsigma)}u(\varsigma) + e^{-\gamma\psi(\varsigma)}u'(\varsigma)$; hence

$$e^{-\gamma\psi(\varsigma)}u'(\varsigma) = (\alpha-1)\psi'(\varsigma)\mathcal{I}_a^{\alpha-1;\psi} f(\varsigma);$$

accordingly,

$$u'(\varsigma) = (\alpha-1)\psi'(\varsigma)e^{\gamma\psi(\varsigma)}\mathcal{I}_a^{\alpha-1;\psi} f(\varsigma). \tag{5}$$

Integrating Equation (5), it follows that

$$u(\varsigma) = u(a) + (\alpha-1)\int_a^\varsigma \psi'(\tau)e^{\gamma\psi(\tau)}\mathcal{I}_a^{\alpha-1;\psi} f(\tau)d\tau,$$

condition $\varphi(a) = 0$ implies $u(a) = 0$; hence

$$\varphi(\varsigma) = (\alpha-1)e^{-\gamma\psi(\varsigma)}\int_a^\varsigma \psi'(\tau)e^{\gamma\psi(\tau)}\mathcal{I}_a^{\alpha-1;\psi} f(\tau)d\tau.$$

This finishes the proof. □

3. Existence Theorems

We always need fixed-point theorems, which are auxiliary tools for dealing with nonlinear differential equations. The idea is to prove that this equation has a fixed point, which is the desired solution. In our research, we rely on two fixed-point theories, BFPT and SFPT [31].

Theorem 2 ([31])**.** *(BFPT). Let H be a Banach space. If $Z : H \to H$ is a contraction, then Z has a unique fixed point in H.*

Theorem 3 ([31])**.** *(SFPT). Let H be a closed, bounded, and convex subset of Banach space X, and the mapping $Z : H \to H$ is a continuous map such that the set $\{z_\varphi : \varphi \in H\}$ is relatively compact. Then, Z has at least one fixed point.*

In view of Theorem 1, we define the operator $\mathcal{P}$ on $\mathcal{C}(J, \mathbb{R})$, as

$$\mathcal{P}\varphi(\varsigma) = (\alpha-1)e^{-\gamma\psi(\varsigma)}\int_a^\varsigma \psi'(\tau)e^{\gamma\psi(\tau)}\mathcal{I}_a^{\alpha-1;\psi} f(\tau)d\tau. \tag{6}$$

Theorem 4. *The operator $\mathcal{P}$ is completely continuous.*

Proof. With our knowledge of the continuity of the function f, from it, we inevitably obtain that continuity of the operator $\mathcal{P}$. Let $\mathcal{B}$ be a bounded proper subset of $\mathcal{C}(J, \mathbb{R})$, so there exists a positive real number A_f, such as $|f(\varsigma, \varphi)| \le A_f$, for any order pair $(\varsigma, \varphi) \in J \times \mathcal{B}$. Additionally,

$$|\mathcal{P}\varphi(\varsigma)| \le \frac{A_f e^{-\gamma\psi(a)}}{\gamma\Gamma(\alpha-1)}(\psi(\Im) - \psi(a))^{\alpha-1}\left(e^{\gamma\psi(\Im)} - e^{\gamma\psi(a)}\right).$$

We apply the maximum over J, and we conclude that the operators $\mathcal{P}$ is bounded on $\mathcal{C}(J, \mathbb{R})$. We are now interested in the proof of the equicontinuity of $\mathcal{P}$. For this, let $a \le \varsigma_1 < \varsigma_2 \le \Im$, then

$$\begin{aligned}
|\mathcal{P}\varphi(\varsigma_2) - \mathcal{P}\varphi(\varsigma_1)| \le{} & \frac{A_f e^{-\gamma\psi(\varsigma_2)}}{\gamma\Gamma(\alpha-1)}(\psi(\Im)-\psi(a))^{\alpha-1}\left(e^{\gamma\psi(\varsigma_2)} - e^{\gamma\psi(\varsigma_1)}\right) \\
& + \frac{A_f\left(e^{-\gamma\psi(\varsigma_2)} - e^{-\gamma\psi(\varsigma_1)}\right)}{\gamma\Gamma(\alpha-1)}(\psi(\Im)-\psi(a))^{\alpha-1}\left(e^{\gamma\psi(\varsigma_1)} - e^{\gamma\psi(a)}\right) \\
& \le \frac{MA_f}{\Gamma(\alpha-1)}(\psi(\Im)-\psi(a))^{\alpha-1}(\psi(\varsigma_2)-\psi(\varsigma_1)) \\
& + \frac{MA_f}{\Gamma(\alpha-1)}(\psi(\Im)-\psi(a))^{\alpha-1}\left(1-e^{\gamma(\psi(a)-\psi(\varsigma_1))}\right)(\psi(\varsigma_2)-\psi(\varsigma_1)) \\
& = \frac{MA_f}{\Gamma(\alpha-1)}(\psi(\Im)-\psi(a))^{\alpha-1}\left(2-e^{\gamma(\psi(a)-\psi(\varsigma_1))}\right)(\psi(\varsigma_2)-\psi(\varsigma_1)),
\end{aligned}$$

where $M = \sup_{\varsigma\in[a,\Im]}\psi'(\varsigma)$. As $|\varsigma_2-\varsigma_1|\to 0$, then $|\mathcal{P}\varphi(\varsigma_2)-\mathcal{P}\varphi(\varsigma_1)|\to 0$. These imply that $\mathcal{P}$ is equicontinuous on J. We conclude from it by using the Arzela-Ascoli theorem that the operator $\mathcal{P}$ is completely continuous. This finishes the proof. □

We state next the so-called SFPT.

Theorem 5 ([31]). *We that assume F is a closed, bounded, and convex subset of a Banach space X, and that the mapping $\Delta : U \to U$ is completely continuous, and so it has a fixed point in F. If we define that subset F of $\mathcal{C}(J,\mathbb{R})$ on which the operator $\mathcal{P}$, as defined by (6), is completely continuous, so the problem (1) has the respective solution.*

Theorem 6. *Let B_f be a positive constant, where*

$$\lim_{\varphi\to 0}\frac{f(\varsigma,\varphi)}{\varphi} \le B_f < \infty,$$

so, the problem (1) has a solution.

Proof. The conditions given impose on us the exist of positive constants ρ_f, such that

$$|f(\varsigma,\varphi)| \le \left(1+B_f\right)\rho_f.$$

Therefore, we define the subset F_f of $\mathcal{C}(J,\mathbb{R})$ as

$$F_f = \left\{\varphi\in\mathcal{C}(J,\mathbb{R}) : |\varphi(\varsigma)| \le \rho_f, \varsigma\in J\right\}.$$

Hence, F_f is a subset of $\mathcal{C}(J,\mathbb{R})$. According to Theorem 4, the operator $\mathcal{P}$ is completely continuous; then, according to SFPT 5, each problem of (1) has a solution. This finishes the proof. □

We demonstrate the EUS to each problem of (1) using the contraction principle based on BFPT.

Theorem 7. *Let f be a Lipschitzian function verifying the condition*

$$|f(\varsigma,\varphi) - f(\varsigma,\chi)| \le C_f|\varphi-\chi|,$$

where $\varsigma\in J$, $\varphi,\chi\in\mathbb{R}$, and $C_f > 0$. So, each of (1) has a unique solution when

$$\vartheta_f = \frac{C_f e^{-\gamma\psi(a)}}{\gamma\Gamma(\alpha-1)}(\psi(\Im)-\psi(a))^{\alpha-1}\left(e^{\gamma\psi(\Im)} - e^{\gamma\psi(a)}\right) < 1.$$

Proof. Since f is continuous, there must be positive constants D_f, such that

$$\max\{|f(\varsigma,0)| : \varsigma \in J\} \leq D_f.$$

We begin by proving that $\mathcal{P}\mathfrak{B}_{r_f} \subset \mathfrak{B}_{r_f}$, where $\mathfrak{B}_{r_f}$ is defined by

$$\mathfrak{B}_{r_f} = \left\{\varphi \in \mathcal{C}(J,\mathbb{R}) : \|\varphi\| \leq r_f\right\},$$

such that r_f is given by

$$r_f \geq \frac{D_f e^{-\gamma\psi(a)}}{\gamma\Gamma(\alpha-1)}(\psi(\Im)-\psi(a))^{\alpha-1}\left(e^{\gamma\psi(\Im)}-e^{\gamma\psi(a)}\right)\left(1-\vartheta_f\right)^{-1}.$$

For doing this, let $\varphi \in \mathfrak{B}_{r_f}$, then

$$\begin{aligned}
|\mathcal{P}\varphi(\varsigma)| &\leq (\alpha-1)e^{-\gamma\psi(a)}\int_a^{\varsigma}\psi'(\tau)e^{\gamma\psi(\tau)}\mathcal{I}_a^{\alpha-1;\psi}(|f(\tau,\varphi(\tau))-f(\tau,0)|+|f(\tau,0)|)d\tau \\
&\leq \frac{\left(C_f\|\varphi\|+D_f\right)(\psi(\Im)-\psi(a))^{\alpha-1}e^{-\gamma\psi(a)}}{\gamma\Gamma(\alpha-1)}\left(e^{\gamma\psi(\Im)}-e^{\gamma\psi(a)}\right) \\
&\leq \frac{D_f(\psi(\Im)-\psi(a))^{\alpha-1}e^{-\gamma\psi(a)}}{\gamma\Gamma(\alpha-1)}\left(e^{\gamma\psi(\Im)}-e^{\gamma\psi(a)}\right)+r_f\vartheta_f \\
&\leq r_f.
\end{aligned}$$

Here, we show the contraction principle. To do this, let $\varphi,\chi \in \mathcal{C}(J,\mathbb{R})$, then

$$\begin{aligned}
|\mathcal{P}\varphi(\varsigma)-\mathcal{P}\chi(\varsigma)| &\leq (\alpha-1)e^{-\gamma\psi(a)}\int_a^{\varsigma}\psi'(\tau)e^{-\gamma\psi(\tau)}\mathcal{I}_a^{\alpha-1;\psi}(|f(\tau,\varphi(\tau))-f(\tau,\chi(\tau))|)d\tau \\
&\leq \frac{C_f e^{-\gamma\psi(a)}}{\gamma\Gamma(\alpha-1)}(\psi(\Im)-\psi(a))^{\alpha-1}\left(e^{\gamma\psi(\Im)}-e^{\gamma\psi(a)}\right)\|\varphi-\chi\| \\
&\leq \vartheta_f\|\varphi-\chi\|.
\end{aligned}$$

As $\vartheta_f < 1$, the contraction principles are satisfied. From BFPT, there exists a unique solution for each problem of (1). End of proof. □

Remark 1. *If $\gamma = 0$ in Equation (1), hence, the integral equation will be*

$$\varphi(\varsigma) = \mathcal{I}_a^{\alpha;\psi}f(\varsigma,\varphi(\varsigma)).$$

Hence, all of the above results will be simpler.

We close our research by giving the following two examples.

4. Examples

4.1. Example 1

Consider the following FBVP,

$$\begin{cases} \left(\mathcal{D}_a^{1.7;\psi}+2\mathcal{D}_a^{0.7;\psi}\right)\varphi(\varsigma) = \frac{1}{3}\arctan\varphi(\varsigma),\ \varsigma \in [a,\Im], \\ \varphi(a) = \varphi'(a) = 0. \end{cases} \tag{7}$$

Let $\psi(\varsigma) = \log\varsigma$; here $\alpha = 1.7, \gamma = 2, a = 1, \Im = e$, and $f(\varsigma,\varphi(\varsigma)) = \frac{1}{3}\arctan\varphi(\varsigma)$. We notice that

$$\lim_{\varphi\to 0}\frac{1}{3}\frac{\arctan\varphi}{\varphi} = \frac{1}{3},$$

and

$$\vartheta_f = 0.820\,34 < 1.$$

So, problem (7) has a unique solution in $\mathcal{C}([a,\Im],\mathbb{R})$. So, we can apply Theorems 6 and 7.

4.2. Example 2

Consider the following FDE,

$$\begin{cases} \left(\mathcal{D}_a^{\frac{6}{5};\psi} + \frac{1}{2}\mathcal{D}_a^{\frac{1}{5};\psi}\right)\varphi(\varsigma) = 2\sqrt{3}\log(\varphi(\varsigma)+1),\ \varsigma \in [a,\Im], \\ \varphi(a) = \varphi'(a) = 0. \end{cases} \tag{8}$$

Let $\psi(\varsigma) = \log\varsigma$; here, $\alpha = \frac{6}{5}, \gamma = \frac{1}{2}, a = 1, \Im = e$, and $f(\varsigma,\varphi(\varsigma)) = 2\sqrt{3}\log(\varphi(\varsigma)+1)$. We notice that

$$\lim_{\varphi\to 0} 2\sqrt{3}\frac{\log(\varphi+1)}{\varphi} = 2\sqrt{3},$$

and

$$\vartheta_f = 0.979\,01 < 1.$$

So, problem (8) has a unique solution in $\mathcal{C}([a,\Im],\mathbb{R})$. So, we can apply Theorems 6 and 7.

5. Conclusions

In this research, we have come up with sufficient conditions to prove the EUS of the FDEs solution in the ψ-CFD for orders $1 < \alpha < 2$. This is achieved with the help of BFPT and SFPT. Since we are interested in the EUS, two examples were given to explain the applicability of previously theoretically theorized theorems. We propose future work to apply the same technique to a special type of pantograph equation and its applications.

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References

1. Ali, M.A.; Kara, H.; Tariboon, J.; Asawasamrit, S.; Budak, H.; Hezenci, F. Some New Simpson's-Formula-Type Inequalities for Twice-Differentiable Convex Functions via Generalized Fractional Operators. *Symmetry* **2021**, *13*, 2249. [CrossRef]
2. Baitiche, Z.A.; Derbazi, C.; Benchohra, M.; Zhou, Y.A. New Class of Coupled Systems of Nonlinear Hyperbolic Partial Fractional Differential Equations in Generalized Banach Spaces Involving the ψ-Caputo Fractional Derivative. *Symmetry* **2021**, *13*, 2412. [CrossRef]
3. Meral, F.; Royston, T.; Magin, R. Fractional calculus in viscoelasticity: An experimental study. *Commun. Nonlinear Sci. Numer. Si. Numer. Simul.* **2010**, *15*, 939–945. [CrossRef]
4. Oldham, K. Fractional differential equations in electrochemistry. *Adv. Eng. Softw.* **2010**, *41*, 9–12. [CrossRef]
5. Balachandran, K.; Matar, M.; Trujillo, J.J. Note on controllability of linear fractional dynamical systems. *J. Control Decision* **2016**, *3*, 267–279. [CrossRef]
6. Kiataramkul, C.; Yukunthorn, W.; Ntouyas, S.K.; Tariboon, J. Sequential Riemann–Liouville and Hadamard–Caputo Fractional Differential Systems with Nonlocal Coupled Fractional Integral Boundary Conditions. *Axioms* **2021**, *10*, 174. [CrossRef]

7. Butzer, P.L.; Kilbas, A.A.; Trujillo, J.J. Fractional calculus in the Mellin setting and Hadamard-type fractiona integrals. *J. Math. Anal. Appl.* **2002**, *269*, 1–27. [CrossRef]
8. Gambo, Y.Y.; Jarad, F.; Baleanu, D.; Abdeljawad, T. On Caputo modification of the Hadamard fractional derivatives. *Adv. Differ. Equ.* **2014**, *2014*, 10. [CrossRef]
9. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications Fractional Differential Equations*; North-Holland Mathematics Studies; Elsevier Science B.V.: Amsterdam, The Netherlands, 2006.
10. Pan, N.; Zhang, B.; Cao, J. Degenerate Kirchhoff-type diffusion problems involving the fractional p-Laplacian. *Nonlinear Anal. Real World Appl.* **2017**, *37*, 56–70. [CrossRef]
11. Tao, F.; Wu, X. Existence and multiplicity of positive solutions for fractional Schrodinger equations with critical growth. *Nonlinear Anal. Real World Appl.* **2017**, *35*, 158–174. [CrossRef]
12. Almeida, R. A Caputo fractional derivative of a function with respect to another function. *Commun. Nonlinear Sci. Numer. Simul.* **2017**, *44*, 460–481. [CrossRef]
13. Ahmad, B.; Matar, M.M.; Ntouyas, S.K. On General Fractional Differential Inclusions with Nonlocal Integral Boundary Conditions. *Differ. Equ. Dyn. Syst.* **2016**, *28*, 241–254. [CrossRef]
14. Boulares, H.; Ardjouni, A.; Laskri, Y. Existence and uniqueness of solutions for nonlinear fractional nabla difference systems with initial conditions. *Fract. Differ. Calc.* **2017**, *7*, 247–263. [CrossRef]
15. Hallaci, A.; Boulares, H.; Ardjouni, A. Existence and uniqueness for delay fractional differential equations with mixed fractional derivatives. *Open J. Math. Anal.* **2020**, *4*, 26–31. [CrossRef]
16. Hallaci, A.; Boulares, H.; Kurulay, M. On the Study of Nonlinear Fractional Differential Equations on Unbounded Interval. *Gen. Lett. Math.* **2018**, *5*, 111–117. [CrossRef]
17. Boulares, H.; Ardjouni, A.; Laskri, Y. Existence and Uniqueness of Solutions to Fractional Order Nonlinear Neutral Differential Equations. *Appl. Math.-Notes* **2018**, *18*, 25–33. Available online: http://www.math.nthu.edu.tw/amen/ (accessed on 20 September 2018).
18. Boulares, H.; Benchaabane, A.; Pakkaranang, N.; Shafqat, R.; Panyanak, B. Qualitative properties of positive solutions of a kind for fractional pantograph problems using technique fixed point theory. *Fractal Fract.* **2022**, *6*, 593. [CrossRef]
19. Ardjouni, A.; Boulares, H.; Laskri, Y. Stability in higher-order nonlinear fractional differential equations. *Acta Comment. Univ. Tartu.* **2018**, *22*, 37–47. [CrossRef]
20. Matar, M. On Existence of positive solution for initial value problem of non linear fractional differential equations of order. *Acta Math. Univ. Comen.* **2015**, *84*, 51–57.
21. Alsaedi, A.; Ntouyas, S.K.; Agarwal, R.P.; Ahmad, B. On Caputo type sequential fractional differential equations with nonlocal integral boundary conditions. *Adv. Differ. Equ.* **2015**, *2015*, 33. [CrossRef]
22. Ahmad, B.; Matar, M.M.; Agarwal, R.P. Existence results for fractional differential equations of arbitrary order with nonlocal integral boundary conditions. *Bound. Value Probl.* **2015**, *2015*, 220. [CrossRef]
23. Ahmad, B.; Nieto, J.J. Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions. *Bound. Value Probl.* **2011**, *2011*, 36. [CrossRef]
24. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives: Theory and Applications*; Gordon and Breach Science: Yverdon, Switzerland, 1993.
25. Salim, A.; Alzabut, J.; Sudsutad, W.; Thaiprayoon, C. On Impulsive Implicit ψ-Caputo Hybrid Fractional Differential Equations with Retardation and Anticipation. *Mathematics* **2022**, *10*, 4821. https://doi.org/10.3390/math10244821 [CrossRef]
26. Atangana, A.; Baleanu, D. New fractional derivatives with non-local and non-sin gular kernel: Theory and application to heat transfer model. *Therm. Sci.* **2016**, *20*, 763–769. [CrossRef]
27. Almeida, R.; Malinowska, A.B.; Monteiro, M.T.T. Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications. *Math. Methods Appl. Sci.* **2018**, *41*, 336–352. [CrossRef]
28. Kilbas, A.A. Hadamard-type fractional calculus. *J. Korean Math. Soc.* **2001**, *38*, 1191–1204.
29. Ahmad, B.; Ntouyas, S.K. On Hadamard fractional integro-differential boundary value problems. *J. Appl. Math. Comput.* **2015**, *47*, 119–131. [CrossRef]
30. Alsaedi, A.; Ntouyas, S.K.; Ahmad, B.; Hobiny, A. Nonlinear Hadamard fractional differential equations with Hadamard type nonlocal non-conserved conditions. *Adv. Differ. Equ.* **2015**, *2015*, 285. [CrossRef]
31. Smart, D.R. *Fixed Point Theorems*; Cambridge University Press: London, UK, 1980.