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The n-Point Composite Fractional Formula for Approximating Riemann–Liouville Integrator

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Abstract: In this paper, we aim to present a novel n-point composite fractional formula for approximating a Riemann–Liouville fractional integral operator. With the use of the definite fractional integral's definition coupled with the generalized Taylor's formula, a novel three-point central fractional formula is established for approximating a Riemann–Liouville fractional integrator. Such a new formula, which emerges clearly from the symmetrical aspects of the proposed numerical approach, is then further extended to formulate an n-point composite fractional formula for approximating the same operator. Several numerical examples are introduced to validate our findings.

Keywords: Richardson extrapolation; Riemann–Liouville fractional derivative and integral; Lagrange interpolating polynomial; Caputo derivative

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1. Introduction

Fractional calculus has many uses in the mathematical modeling of chemical phenomena, physics, technical, and economics. It has contributed in a significant way in developing many topics and implementations in applied mathematics, although there are different definitions of the fractional-order operators, see [1–6]. In recent times, many evolutions in the theory of fractional calculus have been investigated to be employed in many fields of science and engineering. In particular, the fundamental of fractional calculus has been approved as an illustrious mathematical facility used to describe many actual applications [7,8]. As a result, various fractional-order differentiators and integrators have been established and approved by many researchers. It is important to highlight that there are two main operators for fractional-order differentiators; the first one is Caputo's derivative operator with a power law function of convolution of a given function related to a local derivative, whereas the other one is the Riemann–Liouville derivative operator with a power law kernel of convolution [9]. In light of the various views of many mathematicians, the Caputo fractional-order differentiator has confirmed that it is more satisfactory for several real applications than that of the Riemann–Liouville derivative operator [10]. This is due to its suitability for using the assumed initial conditions when the fractional derivatives are taken [11]. Regardless of the best operator among the two former operators, the Riemann–Liouville integrator represents an inverse operator for both. This is because the Caputo differentiator is just a modification for the Riemann–Liouville differentiator [9,11].

The fractional-order integrator supposes that different constructs are not compatible and not constantly equivalent with each other. Actually, the fractional-order integrator is commonly employed for expressing an indefinite integral. In former research, there were only two endeavors attempted to establish a generalization of the fundamental theorem in fractional calculus [12,13]. In particular, different forms of such theorem were outlined in [12], whereas the authors in [13] proposed a more directed scheme into implementation than that of the first reference, but also the fractional definite integral definition had not been yet handled at both. Nor did the formulations, which were introduced in [14,15], outline the definition of the fractional definite integral. It looks as though nobody has formulated such a definition [16,17] except M. Ortigueira and J. Machado in their work in [18], whereby their fractional definite integral definition was set and the fundamental theorem of fractional calculus was consequently analyzed.

In light of the former discussion and based on what is usually carried out in the classical numerical analysis, we aim in this work to derive a novel formula enabling one to approximate a definite fractional integral called the n-point composite fractional formula for approximating a Riemann–Liouville fractional integrator. Thanks to O. Manuel and J. Machado's definition of the definite fractional integral coupled with the generalized Taylor's formula, a new three-point central fractional formula is first derived for approximating the Riemann–Liouville fractional integral operator. Then, this formula is extended to n composite points for formulating our main result in this work.

The remainder of this manuscript is constructed in the following manner. Section 2 aims to recollect some essentials and definitions regarding fractional calculus. Section 3 intends to illustrate the primary results of this work, whereby it will exemplify how we will derive the aimed n-point composite fractional formula for approximating the Riemann–Liouville fractional integrator. Section 4 provides two numerical examples that validate our theoretical outcomes, followed by the last section that outlines the concluding remarks of this work.

2. Preliminaries

In this part, some essential definitions and necessary preliminaries in connection with fractional calculus are recalled. This actually paves the way to our main results later on.

Definition 1 ([19,20])**.** *The fractional Riemann–Liouville integrator of a function $h(t)$ of order $\mu > 0$ can be expressed as:*

$$J^{\mu}h(t) = \frac{1}{\Gamma(\mu)} \int_0^t h(s)(t-s)^{\mu-1}ds, \tag{1}$$

where $t > 0$ and $\mu > 0$.

In what follows, we recall certain properties of the fractional Riemann–Liouville integral operator for completeness:

$$1)\ \ J^0 h(t) = h(t). \tag{2}$$

$$2)\ \ J^{\mu}t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\mu+\gamma+1)}t^{\mu+\gamma},\ \ \gamma \geq -1. \tag{3}$$

$$3)\ \ J^{\mu}J^{\beta}h(t) = J^{\beta}J^{\mu}h(t) \qquad \mu, \beta \geq 0. \tag{4}$$

$$4)\ \ J^{\mu}J^{\beta}h(t) = J^{\mu+\beta}h(t) \qquad \mu, \beta \geq 0. \tag{5}$$

Definition 2 ([19,20])**.** *The Caputo fractional differentiator of a function $h(t)$ of order $\mu > 0$ can be expressed as:*

$$D_t^{\mu}h(t) = \frac{1}{\Gamma(m-\mu)} \int_0^t (t-s)^{m-\mu-1} h^{(m)}(s)\, ds,\ m-1 < \mu \leq m, \tag{6}$$

where $m \in \mathbb{N}$ and $t > 0$.

In the following content, we list some properties of the Caputo differentiator [19,20]:

$$1)\ \ D_t^{\mu} c = 0,\ \text{where } c \text{ is constant.} \tag{7}$$

$$2)\ \ D_t^{\mu} t^{\rho} = \frac{\Gamma(\rho+1)}{\Gamma(\rho-\mu+1)} t^{\rho-\mu},\ \text{where } \rho > \mu - 1. \tag{8}$$

$$3)\ \ D_t^{\mu}(\mu h(t) + \omega g(t)) = \mu D_t^{\mu}(h(t)) + \omega D_t^{\mu}(g(t)), \tag{9}$$

where μ and ω are constant. In the same regard, we report below some other properties related to the composition between the previous two operators [19,20]:

$$D_t^{\alpha} J^{\alpha} h(t) = h(t), \tag{10}$$

and

$$J^{\alpha} D_t^{\alpha} h(t) = h(t) - \sum_{i=1}^{n} h^i(0^+) \frac{t^i}{i!}, \tag{11}$$

where $t > 0$ and $n - 1 < \alpha \leq n$ such that $n \in \mathbb{N}$.

Definition 3 ([19,20])**.** *The Riemann–Liouville fractional differentiator of a function $h(t)$ of order $\mu > 0$ might be outlined in terms of the Riemann–Liouville fractional integrator as:*

$$D^{\mu} h(t) = D^{m}[J^{\rho} h(t)], \tag{12}$$

where $\rho = m - \mu$, $0 < \rho < 1$ and m is the smallest integral greater than μ.

Hereinafter, we recall two highly significant results that help us in deriving the main results of this work; the first one referred to M. Ortigueira and J. Machado, who established a proper formula to find the exact values of given definite fractional integrals [18], while the other one referred to Z. Odibat and S. Momani, who provide a generalization to the well-known Taylor theorem in its fractional case [20].

Definition 4 ([18])**.** *The definite fractional integral of the function f of order α is given by:*

$$J_a^{\alpha} f(x) = \int_a^b f^{(-\alpha+1)}(x).dx = \int_a^b D_a^{-\alpha+1} f(x) dx, \tag{13}$$

where $-\infty < a < b < \infty$ and $\alpha - 1 < n \leq \alpha$ such that $n \in \mathbb{N}$.

Theorem 1 ([20])**.** *(Generalized Taylor's Theorem) Suppose that $D_t^{k\alpha} f(x) \in C^{n+1}(a, b]$ for $k = 0, 1, \cdots, n+1$, where $0 < \alpha \leq 1$. Then, the function f can be expanded about $x = x_0$ as follows:*

$$f(x) = \sum_{i=0}^{n} \frac{(x-x_0)^{i\alpha}}{\Gamma(i\alpha+1)} D_t^{i\alpha} f(x_0) + \frac{(x-x_0)^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)} D_t^{(n+1)\alpha} f(\xi), \tag{14}$$

where $0 < \xi < x$ and $x \in (a, b]$.

3. Main Results

In this section, we aim to present the main results of this work. However, before we begin, we should first recall a very significant result related to the Caputo fractional differentiator. In fact, this result has been recently derived in [21] for the purpose of approximating such a differentiator, D_t^{α}, where $0 < \alpha \leq 1$. It has been derived in light of using a similar manner used in [22], and it has been called the modified three-point fractional formula for approximating the Caputo fractional differentiator.

Theorem 2 ([21]). *Suppose that $f \in C^3[a,b]$ and x_0, x_1, x_2 are distinct points in the interval $[a,b]$ such that $a = x_0 < x_1 = x_0 + h < x_2 = x_0 + 2h = b$ with $h > 0$. Then, the modified three-point fractional formula for approximating the Caputo fractional differentiator can be given by:*

$$
\begin{aligned}
D_t^{\alpha} f(x) =& \frac{x^{2-\alpha}}{h^2\Gamma(3-\alpha)}\Big(f(x_0) - 2f(x_1) + f(x_2)\Big) \\
&- \frac{x^{1-\alpha}}{2h^2\Gamma(2-\alpha)}\Big(f(x_0)(x_1+x_2) - 2f(x_1)(x_0+x_2) + f(x_2)(x_0+x_1)\Big) \\
&+ \frac{f^{(3)}(\xi)}{6}\left(\frac{6}{\Gamma(4-\alpha)}x^{3-\alpha} - \frac{2(x_0+x_1+x_2)}{\Gamma(3-\alpha)}x^{2-\alpha} + \frac{(x_0x_1+x_0x_2+x_1x_2)}{\Gamma(2-\alpha)}x^{1-\alpha}\right),
\end{aligned}
\tag{15}
$$

for a generally unknown $\xi \in (a,b)$, where $x \in [a,b]$.

In consequence of the previous result, we can outline the next result that could help one to approximate the Caputo fractional differentiator $D_t^{2\alpha}$, where $0 < \alpha \le 1$.

Corollary 1. *Suppose that $f \in C^3[a,b]$ and x_0, x_1, x_2 are distinct points in the interval $[a,b]$ such that $a = x_0 < x_1 = x_0 + h < x_2 = x_0 + 2h = b$ with $h > 0$. Let $0 < \alpha \le 1$, then the modified three-point fractional formula for approximating Caputo fractional differentiator $D_t^{2\alpha}$ can be given by:*

$$
D_t^{2\alpha} f(x) = \frac{x^{2-2\alpha}}{h^2\Gamma(3-2\alpha)}(f(x_0) - 2f(x_1) + f(x_2)) + \frac{f^{(3)}(\xi)}{6}\left(\frac{6x^{3-2\alpha}}{\Gamma(4-2\alpha)} - \frac{2(x_0+x_1+x_2)}{\Gamma(3-2\alpha)x^{2-2\alpha}}\right). \tag{16}
$$

for a generally unknown $\xi \in (a,b)$, where $x \in [a,b]$.

Proof. In order to prove this result, we simply apply all properties of the Caputo differentiator reported in (7) on (15). In particular, one can operate D_t^{α} twice again into the modified three-point fractional formula (15) to obtain the desired result. □

Remark 1. *It is worth mentioning that the well-known classical central three-point formulae reported for approximating the first and the second derivatives in [22] can be obtained easily by just substituting $\alpha = 1$ into (15) and (16), respectively.*

In the following content, we aim to derive a new formula for approximating the Riemann–Liouville fractional integral operator J^{α}, which would be, from now on, called the three-point central fractional formula for approximating Riemann–Liouville fractional integrator. This will be carried out by applying on the generalized Taylor's Theorem 1 coupled with considering definite fractional integral's Definition 4.

Theorem 3. *Let $D_t^{n\alpha} f \in C^4[a,b]$ for $n = 0,1,2,3,4$ and $0 < \alpha \le 1$. Suppose $a = x_0 < x_1 = x_0 + h < x_2 = x_0 + 2h = b$ with $h > 0$. Then, the three-point central fractional formula about $x = x_1$ for approximating a Riemann–Liouville fractional integrator is given by:*

$$
\begin{aligned}
J^{\alpha} f(x) = 2hf(x_1) &+ \frac{2h^{3\alpha}x_1^{2-2\alpha}}{\Gamma(3\alpha+1)\Gamma(3-2\alpha)}(f(x_0) - 2f(x_1) + f(x_2)) \\
&+ \frac{2h^{3\alpha}}{6\Gamma(3\alpha+1)}f^{(3)}(\xi)\left(\frac{6x_1^{(3-2\alpha)}}{\Gamma(4-2\alpha)} - \frac{2(x_0+x_1+x_2)}{\Gamma(3-2\alpha)}x_1^{(2-2\alpha)}\right) \\
&+ \frac{2h^{5\alpha}}{\Gamma(5\alpha+1)}D_t^{4\alpha} f(\xi),
\end{aligned}
\tag{17}
$$

for a generally unknown $\xi \in (a,b)$, where $x \in [a,b]$.

Proof. In order to prove this result, we first assume that all the above assumptions hold. Then, by expanding the function f about $x = x_1$ using the generalized Taylor Theorem 1, we obtain:

$$\begin{aligned} f(x) =& f(x_1) + D_t^{\alpha} f(x_1) \frac{(x-x_1)^{\alpha}}{\Gamma(\alpha+1)} + D_t^{2\alpha} f(x_1) \frac{(x-x_1)^{2\alpha}}{\Gamma(2\alpha+1)} \\ &+ D_t^{3\alpha} f(x_1) \frac{(x-x_1)^{3\alpha}}{\Gamma(3\alpha+1)} + D_t^{4\alpha} f(\xi) \frac{(x-x_1)^{4\alpha}}{\Gamma(4\alpha+1)}, \end{aligned} \tag{18}$$

for some $\xi \in (a, b)$. Now, by applying the Riemann–Liouville fractional integral operator to both sides of the above equality, we obtain:

$$\begin{aligned} J^{\alpha} f(x) =& J^{\alpha} f(x_1) + \frac{D_t^{\alpha} f(x_1)}{\Gamma(\alpha+1)} J^{\alpha}[(x-x_1)^{\alpha}] + \frac{D_t^{2\alpha} f(x_1)}{\Gamma(2\alpha+1)} J^{\alpha}\left[(x-x_1)^{2\alpha}\right] \\ &+ \frac{D_t^{3\alpha} f(x_1)}{\Gamma(3\alpha+1)} J^{\alpha}\left[(x-x_1)^{3\alpha}\right] + \frac{D_t^{4\alpha} f(\xi)}{\Gamma(4\alpha+1)} J^{\alpha}\left[(x-x_1)^{4\alpha}\right]. \end{aligned} \tag{19}$$

With the help of using (13) coupled with considering $\rho = -\alpha + 1$, we obtain:

$$\begin{aligned} J^{\alpha} f(x) =& f(x_1) \int_a^b dx + \frac{D_t^{\alpha} f(x_1)}{\Gamma(\alpha+1)} \int_a^b D^{\rho}(x-x_1)^{\alpha} dx + \frac{D_t^{2\alpha} f(x_1)}{\Gamma(2\alpha+1)} \int_a^b D^{\rho}(x-x_1)^{2\alpha} dx \\ &+ \frac{D_t^{3\alpha} f(x_1)}{\Gamma(3\alpha+1)} \int_a^b D^{\rho}(x-x_1)^{3\alpha} dx + \frac{D_t^{4\alpha} f(\xi)}{\Gamma(4\alpha+1)} \int_a^b D^{\rho}(x-x_1)^{4\alpha} dx. \end{aligned} \tag{20}$$

This consequently yields:

$$\begin{aligned} J^{\alpha} f(x) =& 2h f(x_1) + \frac{D_t^{\alpha} f(x_1)}{\Gamma(\alpha+1)} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\rho)} \int_a^b (x-x_1)^{\alpha-\rho} dx \\ &+ \frac{D_t^{2\alpha} f(x_1)}{\Gamma(2\alpha+1)} \frac{\Gamma(2\alpha+1)}{\Gamma(2\alpha+1-\rho)} \int_a^b (x-x_1)^{2\alpha-\rho} dx \\ &+ \frac{D_t^{3\alpha} f(x_1)}{\Gamma(3\alpha+1)} \frac{\Gamma(3\alpha+1)}{\Gamma(3\alpha+1-\rho)} \int_a^b (x-x_1)^{3\alpha-\rho} dx \\ &+ \frac{D_t^{4\alpha} f(\xi)}{\Gamma(4\alpha+1)} \frac{\Gamma(4\alpha+1)}{\Gamma(4\alpha+1-\rho)} \int_a^b (x-x_1)^{4\alpha-\rho} dx. \end{aligned} \tag{21}$$

By simplifying the above equality, we obtain:

$$\begin{aligned} J^{\alpha} f(x) =& 2h f(x_1) + \frac{D_t^{\alpha} f(x_1)}{\Gamma(2\alpha)} \int_a^b (x-x_1)^{2\alpha-1} dx + \frac{D_t^{2\alpha} f(x_1)}{\Gamma(3\alpha)} \int_a^b (x-x_1)^{3\alpha-1} dx \\ &+ \frac{D_t^{3\alpha} f(x_1)}{\Gamma(4\alpha)} \int_a^b (x-x_1)^{4\alpha-1} dx + \frac{D_t^{4\alpha} f(\xi)}{\Gamma(5\alpha)} \int_a^b (x-x_1)^{5\alpha-1} dx, \end{aligned} \tag{22}$$

which immediately means:

$$\begin{aligned} J^{\alpha} f(x) =& 2h f(x_1) + \frac{D_t^{\alpha} f(x_1)}{\Gamma(2\alpha)} \frac{(x-x_1)^{2\alpha}}{(2\alpha)} \bigg|_{x_0}^{x_2} + \frac{D_t^{2\alpha} f(x_1)}{\Gamma(3\alpha)} \frac{(x-x_1)^{3\alpha}}{(3\alpha)} \bigg|_{x_0}^{x_2} \\ &+ \frac{D_t^{3\alpha} f(x_1)}{\Gamma(4\alpha)} \frac{(x-x_1)^{4\alpha}}{(4\alpha)} \bigg|_{x_0}^{x_2} + \frac{D_t^{4\alpha} f(\xi)}{\Gamma(5\alpha)} \frac{(x-x_1)^{5\alpha}}{(5\alpha)} \bigg|_{x_0}^{x_2}. \end{aligned} \tag{23}$$

This implies:

$$
\begin{aligned}
J^{\alpha} f(x) = & 2hf(x_1) + \frac{D_t^{\alpha} f(x_1)}{\Gamma(2\alpha+1)}\left((h)^{2\alpha} - (-h)^{2\alpha}\right) + \frac{D_t^{2\alpha} f(x_1)}{\Gamma(3\alpha+1)}\left((h)^{3\alpha} - (-h)^{3\alpha}\right) \\
& + \frac{D_t^{3\alpha} f(x_1)}{\Gamma(4\alpha+1)}\left((h)^{4\alpha} - (-h)^{4\alpha}\right) + \frac{D_t^{4\alpha} f(\xi)}{\Gamma(5\alpha+1)}\left((h)^{5\alpha} - (-h)^{5\alpha}\right).
\end{aligned} \tag{24}
$$

One might observe that when $\alpha = 1$, the above equality will be defined if $(-h)^{2\alpha} = (h)^{2\alpha}$, $(-h)^{3\alpha} = -(h)^{3\alpha}$, $(-h)^{4\alpha} = (h)^{4\alpha}$, and $(-h)^{5\alpha} = -(h)^{5\alpha}$. Based on this observation, we can assert the following equation:

$$
J^{\alpha} f(x) = 2hf(x_1) + \frac{2h^{3\alpha}}{\Gamma(3\alpha+1)} D_t^{2\alpha} f(x_1) + \frac{2h^{5\alpha}}{\Gamma(5\alpha+1)} D_t^{4\alpha} f(\xi). \tag{25}
$$

Now, by substituting the modified three-point fractional formula for approximating Caputo fractional differentiator $D_t^{2\alpha}$ reported in (16) into (25), we obtain:

$$
\begin{aligned}
J^{\alpha} f(x) = & 2hf(x_1) + \frac{2h^{3\alpha}}{\Gamma(3\alpha+1)}\left[\frac{x_1^{2-2\alpha}}{\Gamma(3-2\alpha)}(f(x_0) - 2f(x_1) + f(x_2))\right. \\
& \left. + \frac{f^{(3)}(\xi)}{6}\left(\frac{6x_1^{(3-2\alpha)}}{\Gamma(4-2\alpha)} - \frac{2(x_0+x_1+x_2)}{\Gamma(3-2\alpha)} x_1^{(2-2\alpha)}\right)\right] + \frac{2h^{5\alpha}}{\Gamma(5\alpha+1)} D_t^{4\alpha} f(\xi),
\end{aligned} \tag{26}
$$

for some $\xi \in (a,b)$. Simplifying the above equation yields:

$$
\begin{aligned}
J^{\alpha} f(x) = & 2hf(x_1) + \frac{2h^{3\alpha} x_1^{2-2\alpha}}{\Gamma(3\alpha+1)\Gamma(3-2\alpha)}(f(x_0) - 2f(x_1) + f(x_2)) \\
& + \frac{2h^{3\alpha}}{6\Gamma(3\alpha+1)} f^{(3)}(\xi)\left(\frac{6x_1^{(3-2\alpha)}}{\Gamma(4-2\alpha)} - \frac{2(x_0+x_1+x_2)}{\Gamma(3-2\alpha)} x_1^{(2-2\alpha)}\right) + \frac{2h^{5\alpha}}{\Gamma(5\alpha+1)} D^{4\alpha} f(\xi),
\end{aligned} \tag{27}
$$

which represents the aimed desired result. □

In light of the previous discussions, one could notice that the error term reported in (27) is too high when one wants to estimate it. From this point of view and to obtain more accuracy in regard to approximating the Riemann–Liouville fractional integrator, we introduce in what followed, one of the main results for this work, the n-point composite fractional formula for approximating the Riemann–Liouville fractional integrator. This would be achieved by tracking the same procedure used to obtain the n-point composite Simpson/trapezoidal formulae [22].

Corollary 2. *Let $D_t^{k\alpha} f \in C^4[a,b]$ for $k = 0,1,2,3,4$ and $0 < \alpha \leq 1$. Suppose $a = x_0 < x_1 = x_0 + h < x_2 = x_0 + 2h < \cdots < x_n = x_0 + nh = b$ with $h > 0$ and $n \geq 2$. Then, the n-point composite fractional formula for approximating the Riemann–Liouville fractional integrator is given by:*

$$
J^{\alpha} f(x) \approx 2h \sum_{i=0}^{\frac{n-2}{2}} f(x_{2i+1}) + \frac{2h^{3\alpha}}{\Gamma(3\alpha+1)\Gamma(3-2\alpha)} \prod_{i=0}^{\frac{n-2}{2}} x_{2i+1}(f(x_{2i}) - 2f(x_{2i+1}) + f(x_{2i+2})), \tag{28}
$$

for some $\xi \in (a,b)$, where $x \in [a,b]$.

Proof. To prove this result, one may rewrite $J^{\alpha} f(x)$ as:

$$
J^{\alpha} f(x) = J_{x_0}^{\alpha} f(x) + J_{x_2}^{\alpha} f(x) + \cdots + J_{x_{n-2}}^{\alpha} f(x). \tag{29}
$$

With the use of (1), the above equality can be re-expressed as:

$$
\begin{aligned}
J^{\alpha}f(x) &= \frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}f(t)dt \\
&= \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}(x-t)^{\alpha-1}f(t) + \frac{1}{\Gamma(\alpha)}\int_{x_2}^{x}(x-t)^{\alpha-1}f(t) + \cdots + \frac{1}{\Gamma(\alpha)}\int_{x_{n-2}}^{x}(x-t)^{\alpha-1}f(t).
\end{aligned} \tag{30}
$$

Now, due to the approximate version of the three-point central fractional formula for approximating the Riemann–Liouville fractional integrator has the form:

$$
J^{\alpha}f(x) \approx 2hf(x_1) + \frac{2h^{3\alpha}x_1^{2-2\alpha}}{\Gamma(3\alpha+1)\Gamma(3-2\alpha)}(f(x_0) - 2f(x_1) + f(x_2)), \tag{31}
$$

then equality (30) with the use of (31) can yield:

$$
\begin{aligned}
J^{\alpha}f(x) \approx & \left[2hf(x_1) + \frac{2h^{3\alpha}x_1^{2-2\alpha}}{\Gamma(3\alpha+1)\Gamma(3-2\alpha)}(f(x_0) - 2f(x_1) + f(x_2))\right] \\
& + \left[2hf(x_3) + \frac{2h^{3\alpha}x_3^{2-2\alpha}}{\Gamma(3\alpha+1)\Gamma(3-2\alpha)}(f(x_2) - 2f(x_3) + f(x_4))\right] \\
& + \left[2hf(x_5) + \frac{2h^{3\alpha}x_5^{2-2\alpha}}{\Gamma(3\alpha+1)\Gamma(3-2\alpha)}(f(x_4) - 2f(x_5) + f(x_6))\right] \\
& + \\
& \vdots \\
& + \left[2hf(x_{n-1}) + \frac{2h^{3\alpha}x_{n-1}^{2-2\alpha}}{\Gamma(3\alpha+1)\Gamma(3-2\alpha)}(f(x_{n-2}) - 2f(x_{n-1}) + f(x_n))\right].
\end{aligned} \tag{32}
$$

Immediately, simplifying the above equation gives:

$$
\begin{aligned}
J^{\alpha}f(x) =& 2h(f(x_1) + f(x_3) + f(x_5) + \cdots + f(x_{n-1})) + \frac{2h^{3\alpha}}{\Gamma(3\alpha+1)\Gamma(3-2\alpha)} \\
& \times \Big[x_1^{2-2\alpha}(f(x_0) - 2f(x_1) + f(x_2)) \\
& \quad + x_3^{2-2\alpha}(f(x_2) - 2f(x_3) + f(x_4)) \\
& \quad + x_5^{2-2\alpha}(f(x_4) - 2f(x_5) + f(x_6)) \\
& \quad + \\
& \quad \vdots \\
& \quad + x_{n-1}^{2-2\alpha}(f(x_{n-2}) - 2f(x_{n-1}) + f(x_n))\Big].
\end{aligned} \tag{33}
$$

In other words, we have the following composite formula:

$$
J^{\alpha}f(x) \approx 2h\sum_{i=0}^{\frac{n-2}{2}} f(x_{2i+1}) + \frac{2h^{3\alpha}}{\Gamma(3\alpha+1)\Gamma(3-2\alpha)}\prod_{i=0}^{\frac{n-2}{2}} x_{2i+1}(f(x_{2i}) - 2f(x_{2i+1}) + f(x_{2i+2})). \tag{34}
$$

□

4. Numerical Examples

In this part, we intend to verify the effectiveness of our final proposed formula; the n-point composite fractional formula for approximating the Riemann–Liouville integrator for two specific functions. Tables are utilized to display and compare the gained findings.

Example 1. *Let us consider the main function* $f(x) = 2x^3 + 8x$*. Suppose the fractional-order has the values* $\alpha = 1, 0.75, 0.5$ *and the interval is* $[0, 2]$ *with* $n = 10$*. Then, applying Riemann–Liouville integrator generates the following cases:*

- *Case 1: When* $\alpha = 1$*, we obtain:*

$$J^1 f(x) = 0.5x^4 + 4x^2. \tag{35}$$

- *Case 2: When* $\alpha = 0.75$*, we obtain:*

$$J^{0.75} f(x) = 0.7234x^{3.75} + 6.0182x^{1.75}. \tag{36}$$

- *Case 3: When* $\alpha = 0.5$*, we obtain:*

$$J^{0.5} f(x) = 1.0316x^{3.5} + 7.7549x^{1.5}. \tag{37}$$

The results from the above expressions (35)–(37) are compared with the approximate values generated by using the n-point composite fractional formula. Accordingly, a numerical comparison is generated for different values of α *and reported in Table 1.*

It is worth mentioning that the average elapsed time (or the CPU time) that is required to give the approximate value for each of the above Riemann–Liouville integrators using the n-point composite fractional formula is 0.096549 s, which is regarded too short in comparison with finding their exact values analytically.

Table 1. A numerical comparison between the exact and approximate values.

$n \setminus \alpha$	0.5	0.75	1
2	22.27105816166	26.16734499344	23.68537600000
4	22.69349110102	26.26331428545	23.69932800000
6	24.03916177943	26.50011822893	23.72599466666
8	27.18307909882	26.96769964223	23.77049600000
10	28.49632000000	26.79576811488	23.84000000000
Exact:	28.69344858658	26.46463660779	24

Example 2. *Consider now the main function is* $f(x) = sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$*, and suppose the fractional-order has the values* $\alpha = 1, 0.65, 0.25$*. The interval here is also chosen as* $[0, 2]$ *with* $n = 10$*. Then, by applying the Riemann–Liouville integrator, we obtain the following cases:*

- *Case 1: When* $\alpha = 1$*, we have:*

$$J^1 f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{\Gamma(2k+2)}{\Gamma(2k+3)} x^{2k+2}. \tag{38}$$

- *Case 2: When* $\alpha = 0.65$*, we have:*

$$J^{0.65} f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{\Gamma(2k+2)}{\Gamma(2k+2.65)} x^{2k+1.65}. \tag{39}$$

- *Case 3: When* $\alpha = 0.45$*, we have:*

$$J^{0.45} f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{\Gamma(2k+2)}{\Gamma(2k+2.45)} x^{2k+1.45}. \tag{40}$$

Consequently, the results from applying the above three expressions (38)–(40) are compared with the approximate values generated by using the n-point composite fractional formula. As a result, another numerical comparison is generated for different values of α *and reported in Table 2.*

Herein, the average elapsed time required to give the approximate value for each of the above Riemann–Liouville integrators using the n-point composite fractional formula is 0.099873 s. This time is also regarded as too short in comparison with finding their exact values analytically.

Table 2. A numerical comparison between the exact and approximate values.

$n \setminus \alpha$	0.45	0.65	1
2	0.988434759968	1.109574041883	1.416657631095
4	1.046664086456	1.129294584780	1.418163344357
6	1.098873318034	1.271316766203	1.420407266983
8	1.156966874475	1.233598909270	1.423035132930
10	1.293240352453	1.306986260607	1.425632059945
Exact:	1.273563291144	1.362300353262	1.416146836547

In light of the previous examples, one might notice that the proposed n-point composite fractional formula can provide a good approximation for the Riemann–Liouville fractional integral operator as compared with the provided exact values. This inference can lead us in the close future to apply such a formula in several applications, especially those applications that are related to differential equations.

5. Conclusions

In this paper, a new numerical formula called the n-point composite fractional formula for approximating Riemann–Liouville fractional integrators has been successfully established. In particular, this formula has been derived based on establishing another novel formula called the three-point central fractional formula for approximating Riemann–Liouville fractional integrators. Two main notions have been used for attaining these results; the generalized Taylor theorem and the definite fractional integral. Several numerical applications, including solving fractional differential equations, partial fractional differential equations and others, have been left to future considerations.

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References

1. Aho, A.V.; Hopcroft, J.E.; Ullman, J.D. *The Design and Analysis of Computer Algorithms*; Addison-Wesley: Reading, MA, USA, 1974.
2. Batiha, I.M.; Obeidat, A.; Alshorm, S.; Alotaibi, A.; Alsubaie, H.; Momani, S.; Albdareen, M.; Zouidi, F.; Eldin, S.M.; Jahanshahi, H. A Numerical Confirmation of a Fractional-Order COVID-19 Model's Efficiency. *Symmetry* **2022**, *14*, 2583. [CrossRef]
3. Batiha, I.M.; El-Khazali, R.; Ababneh, O.Y.; Ouannas, A.; Batyha, R.M.; Momani, S. Optimal design of $PI^{\rho}D^{\mu}$-controller for artificial ventilation systems for COVID-19 patients. *Aims Math.* **2023**, *8*, 657–675. [CrossRef]
4. Williams, W.K.; Vijayakumar, V.; Nisar, K.S.; Shukla, A. Atangana–Baleanu Semilinear Fractional Differential Inclusions with Infinite Delay: Existence and Approximate Controllability. *ASME J. Comput. Nonlinear Dynam.* **2023**, *18*, 021005. [CrossRef]
5. Bose, C.S.V.; Udhayakumar, R. Analysis on the Controllability of Hilfer Fractional Neutral Differential Equations with Almost Sectorial Operators and Infinite Delay via Measure of Noncompactness. *Qual. Theory Dyn. Syst.* **2023**, *22*, 22. [CrossRef]
6. Dineshkumar, C.; Udhayakumar, R.; Vijayakumar, V.; Shukla, A.; Nisar, K.S. Discussion on the Approximate Controllability of Nonlocal Fractional Derivative by Mittag-Leffler Kernel to Stochastic Differential Systems. *Qual. Theory Dyn. Syst.* **2023**, *22*, 27. [CrossRef]
7. Dababneh, A.; Djenina, N.; Ouannas, A.; Grassi, G.; Batiha, I.M.; Jebril, I.H. A New Incommensurate Fractional-Order Discrete COVID-19 Model with Vaccinated Individuals Compartment. *Fractal Fract.* **2022**, *6*, 456. [CrossRef]
8. Ouannas, A.; Batiha, I.M.; Bekiros, S.; Liu, J.; Jahanshahi, H.; Aly, A.A.; Alghtani, A.H. Synchronization of the Glycolysis Reaction-Diffusion Model via Linear Control Law. *Entropy* **2021**, *23*, 1516. [CrossRef] [PubMed]

9. Albadarneh, R.B.; Batiha, I.M.; Adwai, A.; Tahat, N.; Alomari, A.K. Numerical approach of Riemann-Liouville fractional derivative operator. *Int. J. Electr. Comput. Eng.* **2021**, *11*, 5367–5378. [CrossRef]
10. Bezziou, M.; Dahmani, Z.; Jebril, I.; Kaid, M . Caputo-hadamard approach applications: Solvability for an integro-differential problem of lane and emden type. *J. Math. Comput. Sci.* **2021**, *11*, 1629–1649.
11. Albadarneh, R.B.; Batiha, I.; Alomari, A.K.; Tahat, N. Numerical approach for approximating the Caputo fractional-order derivative operator. *Aims Math.* **2021**, *6*, 12743–12756. [CrossRef]
12. Grigoletto, E.C.; de Oliveira, E.C. Fractional Versions of the Fundamental Theorem of Calculus. *Appl. Math.* **2013**, *4*, 23–33. [CrossRef]
13. Tarasov, V.E. Fractional vector calculus and fractional Maxwell's equations. *Ann. Phys.* **2008**, *323*, 2756–2778. [CrossRef]
14. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
15. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives. Theory and Applications*; Gordon and Breach Science Publishers: London, UK, 1993.
16. Machado, J.T.; Mainardi, F.; Kiryakova, V. Fractional Calculus: Quo Vadimus, 495–526. *Fract. Calc. Appl. Anal.* **2015**, *18*. [CrossRef]
17. Machado, J.T.; Mainardi, F.; Kiryakova, V.; Atanacković, T. Fractional Calculus: D'où venos-nous? Que sommes-Nous Où allons-Nous? *Fract. Calc. Appl. Anal.* **2016**, *19*, 1074–1104. [CrossRef]
18. Ortigueira, M.; Machado, J. Fractional Definite Integral. *Fractal Fract.* **2017**, *1*, 2. [CrossRef]
19. Batiha, I.M.; El-Khazali, R.; AlSaedi, A.; Momani, S. The General Solution of Singular Fractional-Order Linear Time-Invariant Continuous Systems with Regular Pencils. *Entropy* **2018**, *20*, 400. [CrossRef] [PubMed]
20. Odibat, M.Z.; Momani, S. An algorithm for the numerical solution of differential equations of fractional order. *J. Appl. Math. Inform.* **2008**, *26*, 15–27.
21. Batiha; Alshorm, I.M.S.; Ouannas, A.; Momani, S.; Ababneh, O.Y.; Albdareen, M. Modified Three-Point Fractional Formulas with Richardson Extrapolation. *Mathematics* **2022**, *10*, 3489. [CrossRef]
22. Burden, R.L.; Faires, J.D. *Numerical Analysis*, 9th ed.; Brooks/Cole: Pacific Grove, CA, USA, 2011.