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Feng-Liu's Approach to Fixed Point Results of Intuitionistic Fuzzy Set-Valued Maps

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Abstract: The applications of non-zero self distance function have recently been discovered in both symmetric and asymmetric spaces. With respect to invariant point results, the available literature reveals that the idea has only been examined for crisp mappings in either symmetric or asymmetric spaces. Hence, the aim of this paper is to introduce the notion of invariant points for non-crisp set-valued mappings in metric-like spaces. To this effect, the technique of κ-contraction and Feng-Liu's approach are combined to establish new versions of intuitionistic fuzzy functional equations. One of the distinguishing ideas of this article is the study of fixed point theorems of intuitionistic fuzzy set-valued mappings without using the conventional Pompeiu–Hausdorff metric. Some of our obtained results are applied to examine their analogues in ordered metric-like spaces endowed with an order and binary relation as well as invariant point results of crisp set-valued mappings. By using a comparative example, it is observed that a few important corresponding notions in the existing literature are complemented, unified and generalized.

Keywords: fixed point; fuzzy set; fuzzy set-valued mapping; intuitionistic fuzzy set; metric-like space; κ-contraction; symmetry

MSC: 46S40; 47H10; 54H25; 34A12; 46J10

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1. Introduction

Fixed point theory is one of the areas of nonlinear functional analysis that unifies topology, analysis and some applied sciences. In this field, the controllability problem is reduced to a fixed point problem for an applicable operator on a suitable space. A fixed point of this operator, if it exists, becomes a solution of the problem under consideration. In symmetric spaces with metric structure, the first corroborative response to a fixed point problem was propounded in 1922 by Banach [1]. Indeed, the Banach contraction principle (BCP) [1] is a reformulation of the successive approximation techniques originally used by some earlier mathematicians, namely Cauchy, Liouville, Picard, Lipschitz, and so on. Presently, various extensions of the BCP abound in the literature. In general, invariant point results in symmetric spaces are extensions of their analogues in asymmetric spaces. In the former context, lately, Romaguera and Tirado [2] presented copies of the well-known Meir-Keeler invariant point theorem. Recently, Pragati et al. [3] introduced the notion of rectangular quasi-partial b-metric spaces, which is in the domain of asymmetric spaces, and studied various Branciari-type functional equations. For other developments and applications of symmetric and asymmetric spaces, the reader can consult [4–6] and some references therein.

About six decades ago, Nadler [7] initiated a multi-valued extension of the BCP. Following [7], numerous invariant point results for set-valued mappings have been presented by many authors. For some of these results, we can refer the reader to Berinde and Berinde [8], Mizoguchi and Takahashi [9], Monairah and Mohammed [10], to mention a few.

The notion of fuzzy sets (Fset) was initiated by Zadeh [11] in 1965 as one of the uncertainty tools to represent mathematical ideas that agree with everyday life. Currently, the primitive concepts of Fset have been modified in different frameworks. Heilpern [12] used the idea of Fset to establish a class of Fset-valued mappings and proved a fixed point theorem for fuzzy contraction mappings which is a fuzzy analogue of the fixed point theorem of Nadler [7]. Thereafter, more than a handful of authors have studied the existence of fixed point of Fset-valued mapping, for example, see [13–17].

In accordance with [11], intuitionistic Fset (IFset) was brought up by Atanassov [18] as an extension of Fset theory. IFset is superior to Fset as it can evaluate both the degree of membership and non-membership. Not long ago, Azam et al. [19] brought in a new technique for analysing fixed point point results using intuitionistic Fset-valued mappings. Later, Azam and Tabassum [20] provided new conditions for the existence of common coincidence points for three intuitionistic Fset-valued mappings. Tabassum et al. [21] presented the idea of common fixed point theorems for intuitionistic Fset-valued mappings in the setting of $(\mathcal{T}, \mathcal{N}, \pi)$-cut set of IFset. Recently, Rehana et al. [16] introduced the ideas of invariant point results of intuitionistic fuzzy mappings in complex-valued b-metric space. Along the line, intuitionistic fuzzy fixed point theorems for sequence of intuitionistic fuzzy mappings on b-metric space was examined by Maysaa et al. [22]. Li et al. [23] established the multigranulation rough set model in intuitionistic fuzzy neighborhood information system produced by the Internet of Things data and discussed the basic properties of the proposed model.

Meanwhile, the study of novel spaces has become a center of attraction within the mathematical domains. In this context, the idea of metric-like spaces, launched by Amini-Harandi [24], is currently being deeply examined. It is pertinent to note that metric-like spaces are otherwise known as dislocated spaces in the literature. An improvement of a fixed point axiom on the aforementioned spaces has been researched by Hitzler and Seda [25] in the scene of logic programming problems. In few applications of logic programming, it is necessary to have nonzero self distances. To handle this requirement, various types of generalized metric spaces, namely, partial metric space, quasi metric space, metric-like spaces have been introduced. For some recent fixed point results in the scene of metric-like spaces, we refer to [26,27] and the citations within.

Among several developments in fuzzy mathematics, much useful effort has been geared to investigate fuzzy copies of the classical fixed point theorems. In this context, employing the results proposed by Jleli and Samet [28] and Feng and Liu [29] approach into consideration in this work, some novel intuitionistic fuzzy fixed point results for nonlinear intuitionistic fuzzy set-valued κ-contractions in the framework of complete metric-like spaces are examined. It is worth noting that one of the great improvements in multivalued fixed point theory is the Feng-Liu's technique with which one can study the existence of fixed points of multivalued operators without using the conventional Hausfdorff metric. From our knowledge based on the surveyed literature, there is no contribution in the existing results regarding the unification of Feng-Liu's and κ-contraction's approach to discuss the existence of fixed point of intuitionistic fuzzy set-valued mappings.

The paper is structured into eight sections. The introduction is contained in Section 1. Section 2 collates some of the fundamental concepts needed in the sequel. In Section 3, the main ideas of the paper are presented. A few special cases of the key findings herein are highlighted in Section 4. Section 5 provides the version of our results in ordered metric-like spaces. Applications in metric-like spaces endowed with binary relations are proposed in Section 6. Some consequences in crisp multivalued mappings are discussed in Section 7. Lastly, concluding remarks are provided in Section 8.

2. Preliminaries

Let $(\aleph, \wp)$ be a metric space. Denoted by $\mathcal{N}(\aleph)$, $\mathcal{C}(\aleph)$, $C_B(\aleph)$ and $\mathcal{K}(\aleph)$, the family of all nonempty subsets of $\aleph$, the collection of all nonempty closed subsets of $\aleph$, the set of all nonempty closed and bounded subsets of $\aleph$ and the class of all nonempty compact subsets of $\aleph$, in that order. For $G, W \in C_B(\aleph)$, the mapping $\aleph_\wp : C_B(\aleph) \times C_B(\aleph) \longrightarrow \mathbb{R}$ defined by

$$\aleph_\wp(G, W) = \max\{\sup_{\jmath \in W} \wp(\jmath, G), \sup_{\ell \in G} \wp(\ell, W)\},$$

where $\wp(\jmath, G) = \inf_{\ell \in G} \wp(\jmath, \ell)$, is called the Hausdorff metric induced by the metric $\wp$.

A point $u \in \aleph$ is a fixed point of a multivalued mapping $F : \aleph \longrightarrow \mathcal{N}(\aleph)$ if $u \in Fu$. A mapping $F : \aleph \longrightarrow C_B(\aleph)$ is called a multivalued contraction if we can find $\lambda \in (0,1)$ such that $\aleph_\wp(F\jmath, F\ell) \leq \lambda \wp(\jmath, \ell)$.

The following result due to Nadler [7] is the first fixed point theorem for multivalued contraction.

Theorem 1 ([7])**.** *Let $(\aleph, \wp)$ be a complete metric space, $F : \aleph \longrightarrow C_B(\aleph)$ be a multivalued contraction . Then F has a fixed point in $\aleph$.*

The first refinement of Theorem 1 without using the Hausdorff metric was presented by Feng and Liu [29]. To highlight their results, we give the next definition: let $b \in (0,1)$ and $\jmath \in \aleph$, then

$$I_b^\jmath = \{\ell \in \Im\jmath : b\wp(\jmath, \ell) \leq \wp(\jmath, \Im\jmath)\},$$

where $\Im : \aleph \longrightarrow \mathcal{C}(\aleph)$ is a multivalued mapping. We also recall that the mapping $g : \aleph \longrightarrow \mathbb{R}$ is called lower semi-continuous (lsc), if for every sequence $\{\jmath_\flat\}_{\flat \in \mathbb{N}} \subset \aleph$ and $u \in \aleph$,

$$\jmath_x \longrightarrow u \implies g(u) \leq \liminf_{x \longrightarrow \infty} g(\jmath_x).$$

Theorem 2 ([29])**.** *Let $(\aleph, \wp)$ be a complete metric space and $\Im : \aleph \longrightarrow \mathcal{C}(\aleph)$ be a multivalued mapping. Suppose that we can find $b, c \in (0,1)$ with $b < c$ such that each $\jmath \in \aleph$, $\ell \in I_b^\jmath$ such that*

$$\wp(\ell, \Im\ell) \leq c\wp(\jmath, \ell),$$

then $\Im$ has a fixed point in $\aleph$ on the criterion that $\jmath \longmapsto \wp(\jmath, F\jmath)$ is lsc.

Klim and Wardowski [30] refined Theorem 2 in the following manner.

Theorem 3 ([30])**.** *Let $(\aleph, \wp)$ be a complete metric space and $\Im : \aleph \longrightarrow \mathcal{C}(\aleph)$ be a multivalued mapping. Suppose that $b \in (0,1)$ and $\hbar : \mathbb{R}_+ \longrightarrow [0, b)$ exist such that*

$$\limsup_{t \longrightarrow s^+} \hbar(t) < b\ ,\ s \in \mathbb{R}_+,$$

and for each $\jmath \in \aleph$, we can find $\ell \in I_b^\jmath$ such that

$$\wp(\ell, \Im\ell) \leq \hbar(\wp(\jmath, \ell))\wp(\jmath, \ell),$$

then $\Im$ has a fixed point in $\aleph$ on the criterion that $\jmath \longmapsto \wp(\jmath, \Im\jmath)$ is lsc.

Theorem 4 ([30])**.** *Let $(\aleph, \wp)$ be a complete metric space and $F : \aleph \longrightarrow \mathcal{K}(\aleph)$ be a multivalued mapping. Suppose that we can find $b \in (0,1)$ and $\hbar : \mathbb{R}_+ \longrightarrow [0,1)$ satisfying*

$$\limsup_{t \longrightarrow s^+} \hbar(t) < 1\ ,\ s, t \in \mathbb{R}_+,$$

and for each $\jmath \in \aleph$, we can find $\ell \in I_1^{\jmath}$ such that

$$\wp(\ell, F\ell) \leq \hbar(\wp(\jmath, \ell))\wp(\jmath, \ell),$$

then F has a fixed point in $\aleph$ on the criterion that $\jmath \longmapsto \wp(\jmath, F\jmath)$ is lsc.

In the literature, we have numerous modifications of Theorems 2 and 3 (see, e.g., [31–33]).

One of the familiar developments of the BCP was launched by Jleli and Samet [28] under the name κ-contraction. We recall this idea accordingly. Let $\Omega = \{\kappa | \kappa : (0, \infty) \longrightarrow (1, \infty)\}$ be the family of functions fulfilling the following criteria:

(κ_1) κ is nondecreasing;
(κ_2) for every sequence $\{t_x\}_{x \in \mathbb{N}} \subset (0, \infty)$, $\lim\limits_{x \longrightarrow \infty} \kappa(t_x) = 1$ if and only if $\lim\limits_{x \longrightarrow \infty} t_x = 0^+$;
(κ_3) we can find $\eta \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim\limits_{t \longrightarrow 0^+} \dfrac{\kappa(\eta) - 1}{t^\eta} = l$.

Let $(\aleph, \wp)$ be a metric space and $\kappa \in \Omega$. A mapping $g : \aleph \longrightarrow \aleph$ is called κ-contraction if we can find $\tau \in (0, 1)$ such that $\jmath, \ell \in \aleph$ with $\wp(g\jmath, g\ell) > 0$,

$$\kappa(\wp(g\jmath, g\ell)) \leq [\kappa(\wp(\jmath, \ell))]^\tau. \tag{1}$$

By invoking various forms of $\kappa \in \Omega$, we can deduce different types of contractions. As an illustration, let $\kappa(t) = e^{\sqrt{t}}$, then $\kappa \in \Omega$ and (1) becomes

$$\wp(g\jmath, g\ell) \leq \tau^2 \wp(\jmath, \ell),$$

$\jmath, \ell \in \aleph$ with $\wp(g\jmath, g\ell) > 0$. In like direction, for $\kappa(t) = e^{\sqrt{te^t}}$, $\kappa \in \Omega$, (1) changes to

$$\frac{\wp(g\jmath, g\ell)}{\wp(\jmath, \ell)} e^{\wp(g\jmath, gy) - \wp(\jmath, \ell)} \leq \tau^2, \tag{2}$$

$\jmath, \ell \in \aleph$ with $\wp(g\jmath, g\ell) > 0$. Apparently, if a mapping g is a BCP, then it satisfies (2). Though, the converse of this statement is not always true (see [28]). Additionally, it is easy to deduce that if g is a κ-contraction, then g is a contractive mapping, that is, $\wp(g\jmath, g\ell) < \wp(\jmath, \ell)$ with $\jmath, \ell \in \aleph$ with $\jmath \neq \ell$. It comes up that every κ-contraction on a metric space is continuous. The following result is one of the refinements of the BCP, using the idea of κ-contraction.

Theorem 5 ([28], Cor.2.1)**.** *Let $(\aleph, \wp)$ be a complete metric space and $g : \aleph \longrightarrow \aleph$ be a point-valued mapping. Suppose that g is a κ-contraction, then g has a unique fixed point in $\aleph$.*

On related path, the notion of κ-contraction was moved to multivalued mapping by Hancer et al. [34]. Let $(\aleph, \wp)$ be a metric space, $F : \aleph \longrightarrow C_B(\aleph)$ be a multivalued mapping and $\kappa \in \Omega$. Then F is called a multivalued κ-contraction if we can find $\tau \in (0, 1)$ such that

$$\kappa(\aleph_\wp(F\jmath_v, F\ell_v)) \leq [\kappa(\wp(\jmath, \ell))]^\tau, \tag{3}$$

where $\jmath, \ell \in \aleph$ with $\aleph_\wp(F\jmath_v, F\ell_v) > 0$. We can see that every multivalued contraction (in the sense of Nadler [7]) is a multivalued κ-contraction with $\kappa(t) = e^{\sqrt{t}}$.

Theorem 6 ([34])**.** *Let $(\aleph, \wp)$ be a complete metric space and $F : \aleph \longrightarrow \mathcal{K}(\aleph)$ be a multivalued κ-contraction. Then F has a fixed point in $\aleph$.*

In ([34], Example 1), it has been demonstrated that $\mathcal{K}(\aleph)$ cannot be replaced with $C_B(\aleph)$ in Theorem 6. Though, we can investigate $C_B(\aleph)$ in place of $\mathcal{K}(\aleph)$ by affixing on κ, the criterion:

(κ_4) $\kappa(\inf A) = \inf \kappa(A)$ with $A \subset (0, \infty)$ and $\inf A > 0$.

Notice that if κ satisfies (κ_1), then it satisfies (κ_4) if and only if it is right continuous. Let

$$\mho = \{\kappa | \kappa : (0, \infty) \longrightarrow (1, \infty) \text{ satisfying } (\kappa_1) - (\kappa_4)\}.$$

Theorem 7 ([34]). *Let $(\aleph, \wp)$ be a complete metric space and $F : \aleph \longrightarrow C_B(\aleph)$ be a multivalued κ-contraction. Suppose that $\kappa \in \mho$, then F has a fixed point in $\aleph$.*

Recollect that an ordinary subset A of $\aleph$ is described by its characteristic function χ_A, given by $\chi_A : \aleph \longrightarrow \{0, 1\}$ such that

$$\chi_A(\jmath) = \begin{cases} 1, & \text{for } \jmath \in A \\ 0, & \text{for } \jmath \notin A. \end{cases}$$

The value $\chi_A(\jmath)$ indicates if a point is in A or not. This observation is used to set up Fset by allowing the element $\jmath \in A$ to take value in the interval $[0, 1]$. Hence, an Fset in $\aleph$ is a function with domain $\aleph$ and range in $[0, 1] = I$. The class of all Fset in $\aleph$ is denoted by $I^{\aleph}$. Suppose that A is an Fset in $\aleph$, then the function value $A(\jmath)$ is called the grade of membership of $\jmath$ in A. The π-level set of an Fset A is denoted by $[A]_\pi$ and is given as

$$[A]_\pi = \begin{cases} \overline{\{\jmath \in \aleph : A_F(\jmath) > 0\}}, & \text{if } \pi = 0 \\ \{\jmath \in \aleph : A_F(\jmath) \geq \pi\}, & \text{if } \pi \in (0, 1]. \end{cases}$$

Definition 1 ([12]). *Let $\aleph$ be a nonempty set. The mapping $\Xi : \aleph \longrightarrow I^{\aleph}$ is called an Fset-valued mapping. A point $u \in \aleph$ is termed a fuzzy fixed point of Ξ if we can find a $\pi \in (0, 1]$ such that $u \in [\Xi u]_\pi$.*

Definition 2 ([18]). *Let $\aleph$ be a nonempty set. An intuitionistic Fset ϖ in $\aleph$ is a set of ordered triples given by*

$$\varpi = \{\langle \jmath, \mu_\varpi(\jmath), \nu_\varpi(\jmath)\rangle : \jmath \in \aleph\},$$

where $\mu_\varpi : \aleph \longrightarrow [0, 1]$ and $\nu_\varpi : \aleph \longrightarrow [0, 1]$ represent the degrees of membership and non-membership, individually of $\jmath$ in $\aleph$ and satisfy $0 \leq \mu_\varpi + \nu_\varpi \leq 1$, for each $\jmath \in \aleph$. Accordingly, the degree of hesitancy of $\jmath \in \varpi$ is given by

$$h_\varpi(\jmath) = 1 - \mu_\varpi(\jmath) - \nu_\varpi(\jmath).$$

In particular, if $h_\varpi(\jmath) = 0$ with $\jmath \in \aleph$, then an IFset reduces to an Fset.

We depict the set of all intuitionistic Fsets in $\aleph$ by $(IFset)^{\aleph}$.

Definition 3 ([18]). *Let ϖ be an IFset in $\aleph$. Then the π-level set of ϖ is a crisp subset of $\aleph$ denoted by $[\varpi]_\pi$ and is defined as*

$$[\varpi]_\pi = \{\jmath \in \aleph : \mu_\varpi(\jmath) \geq \pi \text{ and } \nu_\varpi(\jmath) \leq 1 - \pi\}, \text{ if } \pi \in [0, 1].$$

Definition 4 ([19]). *Let $L = \{(\pi, \varsigma) : \pi + \varsigma \leq 1,\ (\pi, \varsigma) \in (0, 1] \times [0, 1)\}$ and ϖ is an IFset in $\aleph$. Then the (π, ς)-level set of ϖ is given as*

$$[\varpi]_{(\pi,\varsigma)} = \{\jmath \in \aleph : \mu_\varpi(\jmath) \geq \pi \text{ and } \nu_\varpi(\jmath) \leq \varsigma\}.$$

Example 1. *Let $\aleph = \{\jmath_1, \jmath_2, \jmath_3, \jmath_4, \jmath_5\}$ and ϖ be an IFset in $\aleph$ set up by*

$$\varpi = \{(\jmath_1, 0.6, 0.2), (\jmath_2, 0.5, 0.4), (\jmath_3, 0.1, 0.7), (\jmath_4, 0.3, 0.5), (\jmath_5, 0.4, 0.3)\}.$$

Then the (π, ς)-level sets of ϖ are given by

$[\varpi]_{(0.4,0.3)} = \{\jmath_1, \jmath_5\}$.

$[\varpi]_{(0.1,0.7)} = \{\jmath_1, \jmath_2, \jmath_3, \jmath_4, \jmath_5\}.$
$[\varpi]_{(0.3,0.5)} = \{\jmath_1, \jmath_2, \jmath_4, \jmath_5\}.$

Definition 5 ([19])**.** *Let $\aleph$ be a nonempty set. The mapping $\Theta = \langle \mu_\Theta, \nu_\Theta \rangle : \aleph \longrightarrow (IFset)^{\aleph}$ is called an intuitionistic Fset-valued mapping. A point $u \in \aleph$ is an intuitionistic fuzzy fixed point of Θ if we can find $(\pi, \varsigma) \in (0,1] \times [0,1)$ such that $u \in [\Theta u]_{(\pi,\varsigma)}$.*

Remark 1 ([19])**.** *Suppose that ϖ is an IFset, then the set $\hat{\varpi}$ is set up as*

$$\hat{\varpi} = \{\jmath \in \aleph : \mu_\varpi(\jmath) = \max_{\ell \in \aleph} \mu_\varpi(\ell) \text{ and } \nu_\varpi(\jmath) = \min_{\ell \in \aleph} \nu_\varpi(\ell)\}.$$

We now record some needed concepts of metric-like spaces as follows.

Definition 6 ([24])**.** *Let $\aleph$ be a nonempty set and $\varrho : \aleph \times \aleph \longrightarrow \mathbb{R}_+$ be a mapping satisfying*

(i) *if $\varrho(\jmath, \ell) = 0$, then $\jmath = \ell$;*
(ii) *$\varrho(\jmath, \ell) = \varrho(\ell, \jmath)$;*
(iii) *$\varrho(\jmath, z) \leq \varrho(\jmath, \ell) + \varrho(\ell, z)$,*

with $\jmath, \ell, z \in \aleph$. The mapping ϱ is then called a metric-like on $\aleph$ and $(\aleph, \varrho)$ is called a metric-like space.

Note that in a metric-like space $(\aleph, \varrho)$, the distance $\varrho(\ell, \ell)$ is not always zero.

Definition 7 ([24])**.** *Let $(\aleph, \varrho)$ be a metric-like space. Then, a sequence $\{\jmath_\flat\}_{\flat \in \mathbb{N}}$ in $\aleph$ is said to be*

(i) *ϱ-convergent to a limit u in $\aleph$, if*

$$\lim_{x \longrightarrow \infty} \varrho(\jmath_x, u) = \varrho(u, u).$$

(ii) *ϱ-Cauchy, if $\lim_{x,y \longrightarrow \infty} \varrho(\jmath_x, \jmath_y)$ exists and is finite.*

(iii) *ϱ-complete if for every ϱ-Cauchy sequence $\{\jmath_\flat\}_{\flat \in \mathbb{N}}$, we can find $u \in \aleph$ such that*

$$\lim_{x,y \longrightarrow \infty} \varrho(\jmath_x, \jmath_y) = \varrho(u, u) = \lim_{x \longrightarrow \infty} \varrho(\jmath_x, u).$$

It is pertinent to note that in metric-like spaces, the limit of a ϱ-convergent sequence is not necessarily unique [e.g., see ([24], Remark 2.3)]. These spaces have been examined as a refinement of partial metric space (see [35]). It has been shown that every partial metric space is a metric-like space, but the converse is not always true [e.g., see ([24], Example 2.2)].

Given two metric-like spaces $(\aleph, \varrho)$ and (Y, μ), a function $g : \aleph \longrightarrow Y$ is continuous if

$$\lim_{x \longrightarrow \infty} \varrho(\jmath_x, u) = \lim_{x \longrightarrow \infty} \mu(g(\jmath_x), g(u)).$$

3. Main Results

Let $(\aleph, \varrho)$ be a metric-like space and $\Theta : \aleph \longrightarrow (IFset)^{\aleph}$ be an intuitionistic Fset-valued mapping. For each $\jmath \in \aleph$ and $s \in (0,1]$, consider the set $\kappa_s^{\jmath} \subseteq \aleph$ and the subcollections $(IFset)^{\mathcal{K}(\aleph)}$, $(IFset)^{\mathcal{C}(\aleph)}$ of $(IFset)^{\aleph}$ defined as follows:

$$\kappa_s^{\jmath} = \{\ell \in [\Theta(\jmath)]_{(\pi,\varsigma)} : [\kappa(\varrho(\jmath, \ell))]^s \leq \kappa(\varrho(\jmath, [\Theta(\jmath)]_{(\pi,\varsigma)})), \text{ for each } (\pi, \varsigma) \in (0,1] \times [0,1)\}.$$

$$(IFset)^{\mathcal{K}(\aleph)} = \{\varpi \in (IFset)^{\aleph} : [\varpi]_{(\pi,\varsigma)} \in \mathcal{K}(\aleph), \text{ for each } (\pi, \varsigma) \in (0,1] \times [0,1)\}.$$

$$(IFset)^{\mathcal{C}(\aleph)} = \{\varpi \in (IFset)^{\aleph} : [\varpi]_{(\pi,\varsigma)} \in \mathcal{C}_{\mathcal{B}}(\aleph), \text{ for each } (\pi, \varsigma) \in (0,1] \times [0,1)\}.$$

Remark 2. *For the set $\kappa_s^{\jmath}$, we examine the following cases:*

Case 1. Suppose that $\Theta : \aleph \longrightarrow (IFset)^{\mathcal{K}(\aleph)}$, then $\kappa_s^{\jmath} \neq \emptyset$, $s \in (0,1]$ and $\jmath \in \aleph$ with $\varrho(\jmath, [\Theta\jmath]_{(\pi,\varsigma)}) > 0$ for some $(\pi,\varsigma) \in (0,1] \times [0,1)$. In fact, since $[\Theta\jmath]_{(\pi,\varsigma)} \in \mathcal{K}(\aleph)$, we have $\ell \in [\Theta\jmath]_{(\pi,\varsigma)}$ such that $\varrho(\jmath,\ell) = \varrho(\jmath, [\Theta\jmath]_{(\pi,\varsigma)})$ for each $\jmath \in \aleph$. Hence, $\kappa(\varrho(\jmath,\ell)) = \kappa(\varrho(\jmath, [\Theta\jmath]_{(\pi,\varsigma)}))$. Hence, $\ell \in \kappa_s^{\jmath}$ with $s \in (0,1]$.

Case 2. Suppose that $\Theta : \aleph \longrightarrow (IFset)^{\mathcal{C}(\aleph)}$, then $\kappa_s^{\jmath}$ may contain nothing for some $\jmath \in \aleph$ and $s \in (0,1]$. To understand this, let $\kappa(t) = e^{\sqrt{\frac{t}{10}}}$ for $0 < t \leq 10$ and $\kappa(t) = 36t$ for $t > 10$. Very apparent, $\kappa \in \Omega$. Take $\aleph = [0,\infty)$ and $\varrho : \aleph \times \aleph \longrightarrow \mathbb{R}$ be set up as

$$\varrho(\jmath,\ell) = \begin{cases} 10, & \text{if } \jmath,\ell \in [0,3] \\ \jmath^2 + \ell^2 + 10, & \text{if one of } \jmath,\ell \notin [3,\infty). \end{cases}$$

Then $(\aleph,\varrho)$ is a metric-like space. Notice that ϱ is not a metric on $\aleph$, since $\varrho(0,0) = 10$. Let $\lambda, \zeta \in (0,1]$ and define an intuitionistic Fset-valued mapping $\mathcal{W} : \aleph \longrightarrow (IFset)^{\aleph}$ as follows:

$$\mu_{\mathcal{W}(\jmath)}(t) = \begin{cases} \frac{\lambda}{9}, & \text{if } t \in [3,5] \\ \frac{\lambda}{17}, & \text{if } t \in (5,20] \\ \frac{\lambda}{30}, & \text{if } t \in (20,30] \\ 0, & \text{if } t \notin [3,30], \end{cases} \quad \nu_{\mathcal{W}(\jmath)}(t) = \begin{cases} 0, & \text{if } t \in [3,5] \\ \frac{\zeta}{14}, & \text{if } t \in (5,20] \\ \frac{\zeta}{5}, & \text{if } t \in (20,30] \\ \zeta, & \text{if } t \notin [3,30]. \end{cases}$$

Now, define $\Theta : \aleph \longrightarrow (IFset)^{\aleph}$ as follows:

$$\mu_{\Theta(\jmath)}(t) = \begin{cases} \digamma_{\{0\}}, & \text{if } \jmath = 0 \\ \mu_{\mathcal{W}(\jmath)}, & \text{if } \jmath > 0. \end{cases} \quad \nu_{\Theta(\jmath)}(t) = \begin{cases} \daleth_{\{0\}}, & \text{if } \jmath = 0 \\ \nu_{\mathcal{W}(\jmath)}, & \text{if } \jmath > 0. \end{cases}$$

Suppose that $(\pi,\varsigma) = \left(\frac{\lambda}{9}, 0\right)$, then

$$[\Theta(\jmath)]_{\left(\frac{\lambda}{9},0\right)} = \begin{cases} \{0\}, & \text{if } \jmath = 0 \\ [3\jmath, 5\jmath], & \text{if } \jmath > 0. \end{cases}$$

For $\jmath = 1$, we have $[\Theta(1)]_{\left(\frac{\lambda}{9},0\right)} = [3,5]$. Obviously, $1 \notin [3,5]$ and

$$\begin{aligned} \varrho(1, [\Theta(1)]_{\left(\frac{\lambda}{9},0\right)}) &= \inf\{\varrho(1,\ell) : \ell \in [3,5]\} \\ &= \varrho(1,3) = 10. \end{aligned}$$

Hence, for $s = \frac{1}{2} \in (0,1]$, we have

$$\begin{aligned} \kappa_{\frac{1}{2}}^{1} &= \{\ell \in [\Theta(1)]_{\left(\frac{\lambda}{9},\frac{\zeta}{6}\right)} : [\kappa(\varrho(1,\ell))]^{\frac{1}{2}} \leq \kappa(\varrho(1, [\Theta1]_{\left(\frac{\lambda}{9},\frac{\zeta}{6}\right)}))\} \\ &= \{\ell \in [3,5] : [\kappa(\ell^2 + 10)]^{\frac{1}{2}} \leq \kappa(10)\} \\ &= \{\ell \in [3,5] : 6\sqrt{\ell^2 + 10} \leq e\} \\ &= \emptyset. \end{aligned}$$

Case 3. Suppose that $\Theta : \aleph \longrightarrow (IFset)^{\mathcal{C}(\aleph)}$ and $\kappa \in \mho$, then $\kappa_s^{\jmath} \neq \emptyset$ with $s \in (0,1)$ and $\jmath \in \aleph$ with $\varrho(\jmath, [\Theta\jmath]_{(\pi,\varsigma)}) > 0$. To see this, given that κ is right continuous, there is $\varsigma > 1$ such that

$$\kappa(\varsigma(\varrho(\jmath, [\Theta\jmath]_{(\pi,\varsigma)}))) \leq [\kappa(\varrho(\jmath, [\Theta\jmath]_{(\pi,\varsigma)}))]^{\frac{1}{s}}.$$

Since $\varsigma > 1$, we can find $\ell \in [\Theta\jmath]_{(\pi,\varsigma)}$ such that $\varrho(\jmath,\ell) \leq \varsigma\varrho(\jmath,[\Theta\jmath]_{(\pi,\varsigma)})$. Hence, from (κ_1), we get

$$\begin{aligned}\kappa(\varrho(\jmath,\ell)) &\leq \kappa(\varsigma\varrho(\jmath,[\Theta\jmath]_{(\pi,\varsigma)})) \\ &\leq [\kappa(\varrho(\jmath,[\Theta\jmath]_{(\pi,\varsigma)}))]^{\frac{1}{s}},\end{aligned}$$

from which it follows that $[\kappa(\varrho(\jmath,\ell))]^s \leq \kappa(\varrho(\jmath,[\Theta\jmath]_{(\pi,\varsigma)}))$; that is, $\ell \in \kappa_s^\jmath$.

Definition 8. *Let $(\aleph,\varrho)$ be a metric-like space, $\Theta : \aleph \longrightarrow (IFset)^{\mathcal{C}(\aleph)}$ be an intuitionistic Fset-valued mapping and $\kappa \in \mho$. Then Θ is called a nonlinear intuitionistic Fset-valued κ-contraction of type(A), if there exist $s \in (0,1)$ and $\xi : \mathbb{R}_+ \longrightarrow [0,s)$ such that*

$$\limsup_{t\longrightarrow\varsigma^+} \xi(t) < s \text{ for all } \varsigma \in \mathbb{R}_+, \tag{4}$$

and for any $\jmath \in \aleph$ with $\varrho(\jmath,[\Theta\jmath]_{(\pi,\varsigma)}) > 0$, we can find $\ell \in \kappa_s^\jmath$ such that

$$\kappa(\varrho(\ell,[\Theta\ell]_{(\pi,\varsigma)})) \leq [\kappa(\varrho(\jmath,\ell))]^{\xi(\varrho(\jmath,\ell))}. \tag{5}$$

Definition 9. *Let $(\aleph,\varrho)$ be a metric-like space, $\Theta : \aleph \longrightarrow (IFset)^{\mathcal{K}(\aleph)}$ be an intuitionistic Fset-valued mapping and $\kappa \in \Omega$. Then Θ is called a nonlinear intuitionistic Fset-valued κ-contraction of type(B), if there exists $\xi : \mathbb{R}_+ \longrightarrow [0,1)$ such that*

$$\limsup_{t\longrightarrow\varsigma^+} \xi(t) < 1 \text{ for all } \varsigma \in \mathbb{R}_+, \tag{6}$$

and for any $\jmath \in \aleph$ with $\varrho(\jmath,[\Theta\jmath]_{(\pi,\varsigma)}) > 0$, we can find $\ell \in \kappa_1^\jmath$ such that

$$\kappa(\varrho(\ell,[\Theta\ell]_{(\pi,\varsigma)})) \leq [\kappa(\varrho(\jmath,\ell))]^{\xi(\varrho(\jmath,\ell))}. \tag{7}$$

Theorem 8. *Let $(\aleph,\varrho)$ be a complete metric-like space and $\Theta : \aleph \longrightarrow (IFset)^{\mathcal{C}(\aleph)}$ be an intuitionistic Fset-valued mapping. Suppose that Θ is a nonlinear intuitionistic Fset-valued κ-contraction of type(A), then Θ possesses intuitionistic fuzzy fixed point in $\aleph$ on the condition that $\jmath \longmapsto \varrho(\jmath,[\Theta\jmath]_{(\pi,\varsigma)}$ is lsc.*

Proof. Assume that Θ has no intuitionistic fuzzy fixed point in $\aleph$. Then, with $\jmath \in \aleph$ and $(\pi,\varsigma) \in (0,1] \times [0,1)$, $\varrho(\jmath,[\Theta\jmath]_{(\pi,\varsigma)}) > 0$. Suppose that $[\Theta\jmath]_{(\pi,\varsigma)} \in \mathcal{C}_{\mathcal{B}}(\aleph)$ for each $\jmath \in \aleph$ and $\kappa \in \mho$, by Case 3 of Remark 2, $\kappa_s^\jmath$ is nonempty with $s \in (0,1)$. Now, for any initial point $\jmath_0$, we can find $\jmath_1 \in \kappa_s^{\jmath_0}$ such that

$$\kappa(\varrho(\jmath_1,[\Theta\jmath_1]_{(\pi,\varsigma)})) \leq [\kappa(\varrho(\jmath_0,\jmath_1))]^{\xi(\varrho(\jmath_0,\jmath_1))},$$

and for $\jmath_1 \in \aleph$, we can find $\jmath_2 \in \kappa_s^{\jmath_1}$ such that

$$\kappa(\varrho(\jmath_2,[\Theta\jmath_2]_{(\pi,\varsigma)})) \leq [\kappa(\varrho(\jmath_1,\jmath_2))]^{\xi(\varrho(\jmath_1,\jmath_2))}.$$

By continuing in this fashion, we generate a sequence $\{\jmath_\flat\}_{\flat\in\mathbb{N}}$ in $(\aleph,\varrho)$ with $\jmath_{x+1} \in \kappa_s^{\jmath_x}$ such that

$$\kappa(\varrho(\jmath_{x+1},[\Theta\jmath_{x+1}]_{(\pi,\varsigma)})) \leq [\kappa(\varrho(\jmath_x,\jmath_{x+1}))]^{\xi(\varrho(\jmath_x,\jmath_{x+1}))}, x = 0,1,2,\cdots \tag{8}$$

Next, we demostrate that $\{\jmath_\flat\}_{\flat\in\mathbb{N}}$ is a Cauchy sequence in $\aleph$. Since $\jmath_{x+1} \in \kappa_s^{\jmath_x}$, we get

$$[\kappa(\varrho(\jmath_x,\jmath_{x+1}))]^s \leq \kappa(\varrho(\jmath_x,[\Theta\jmath_x]_{(\pi,\varsigma)})). \tag{9}$$

From (8) and (9), we have

$$\kappa(\varrho(\jmath_{x+1}, [\Theta\jmath_{x+1}]_{(\pi,\varsigma)})) \leq [\kappa(\varrho(\jmath_x, [\Theta_x]_{(\pi,\varsigma)})]^{\frac{\xi(\varrho(\jmath_x,\jmath_{x+1}))}{s}} \tag{10}$$

and

$$\kappa(\varrho(\jmath_{x+1}, \jmath_{x+2})) \leq [\kappa(\varrho(\jmath_x, \jmath_{x+1}))]^{\frac{\xi(\varrho(\jmath_x,\jmath_{x+1}))}{s}}. \tag{11}$$

Using (10), (11) and (κ_1), we see that $\{\varrho(\jmath_x, [\Theta\jmath_x]_{(\pi,\varsigma)})\}_{x\in\mathbb{N}}$ and $\{\varrho(\jmath_x, \jmath_{x+1})\}_{x\in\mathbb{N}}$ are nonincreasing sequences and hence convergent. From (4), we can find $\gamma \in [0,s)$ such that $\lim\limits_{x\longrightarrow\infty} \xi(\varrho(\jmath_x, \jmath_{x+1})) = \gamma$. Hence, we can find $b \in (\gamma, s)$ and $x_0 \in \mathbb{N}$ such that $\xi(\varrho(\jmath_x, \jmath_{x+1})) < b$ with $x \geq x_0$. Hence, using (11), we find that with $x \geq x_0$,

$$\begin{aligned}
1 <\ & \kappa(\varrho(\jmath_x, \jmath_{x+1})) \\
\leq\ & [\kappa(\varrho(\jmath_{x-1}, \jmath_x))]^{\frac{\xi(\varrho(\jmath_{x-1},\jmath_x))}{s}} \\
\leq\ & [\kappa(\varrho(\jmath_{x-2}, \jmath_{x-1}))]^{\frac{\xi(\varrho(\jmath_{x-2},\jmath_{x-1}))}{s}\frac{\xi(\varrho(\jmath_{x-1},\jmath_x))}{s}} \\
&\vdots \\
\leq\ & [\kappa(\varrho(\jmath_0, \jmath_1))]^{\frac{\xi(\varrho(\jmath_0,\jmath_1))}{s}\cdots\frac{\xi(\varrho(\jmath_{x-2},\jmath_{x-1}))}{s}\frac{\xi(\varrho(\jmath_{x-1},\jmath_x))}{s}} \\
\leq\ & [\kappa(\varrho(\jmath_0, \jmath_1))]^{\frac{\xi(\varrho(\jmath_0,\jmath_1))}{s}\cdots\frac{\xi(\varrho(\jmath_{x_0-1},\jmath_{x_0}))}{s}\frac{\xi(\varrho(\jmath_{x_0},\jmath_{x_0+1}))}{s}\cdots\frac{\xi(\varrho(\jmath_{x-2},\jmath_{x-1}))}{s}\frac{\xi(\varrho(\jmath_{x-1},\jmath_x))}{s}} \\
\leq\ & [\kappa(\varrho(\jmath_0, \jmath_1))]^{\frac{\xi(\varrho(\jmath_{x_0},\jmath_{x_0+1}))}{s}\cdots\frac{\xi(\varrho(\jmath_{x-2},\jmath_{x-1}))}{s}\frac{\xi(\varrho(\jmath_{x-1},\jmath_x))}{s}} \\
\leq\ & [\kappa(\varrho(\jmath_0, \jmath_1))]^{\frac{b^{(x-x_0)}}{s^{(x-x_0)}}}.
\end{aligned}$$

Hence, with $x \geq x_0$,

$$1 < \kappa(\varrho(\jmath_x, \jmath_{x+1})) \leq [\kappa(\varrho(\jmath_0, \jmath_1))]^{\left(\frac{b}{s}\right)^{(x-x_0)}}. \tag{12}$$

Since $\lim\limits_{x\longrightarrow\infty} \left(\dfrac{b}{s}\right)^{(x-x_0)} = 0$, then, as $x \longrightarrow \infty$ in (12), we have

$$\lim_{x\longrightarrow\infty} \kappa(\varrho(\jmath_x, \jmath_{x+1})) = 1. \tag{13}$$

Hence, from (κ_2), $\lim\limits_{x\longrightarrow\infty} \varrho(\jmath_x, \jmath_{x+1}) = 0^+$, and from (κ_3), it follows that we can find $\eta \in (0,1)$ and $l \in (0,\infty]$ such that

$$\lim_{x\longrightarrow\infty} \frac{\kappa(\varrho(\jmath_x, \jmath_{x+1}))}{[\varrho(\jmath_x, \jmath_{x+1})]^\eta} = l. \tag{14}$$

From (14), we consider the following cases:

Case 1. $l = \infty$. For this, let $\delta = \frac{l}{2} > 0$. From the definition of limit, we can find $x_1 \in \mathbb{N}$ such that all $x \geq x_1$,

$$\left| \frac{\kappa(\varrho(\jmath_x, \jmath_{x+1})) - 1}{[\varrho(\jmath_x, \jmath_{x+1})]^\eta} - l \right| \leq \delta,$$

from which we have $\frac{\kappa(\varrho(\jmath_x,\jmath_{x+1}))-1}{[\varrho(\jmath_x,\jmath_{x+1})]^\eta} - l \geq l - \delta = \delta$. Then, with $x \geq x_1$ and $\rho = \frac{1}{\delta}$,

$$x[\varrho(\jmath_x, \jmath_{x+1})]^\eta \leq \rho x[\kappa(\varrho(\jmath_x, \jmath_{x+1}))] - 1.$$

Case 2. $l = \infty$. Let $\delta > 0$ be an arbitrary positive number. In this case, we can find $x_1 \in \mathbb{N}$ such that all $x \geq x_1$, $\frac{\kappa(\varrho(\jmath_x,\jmath_{x+1}))}{[\varrho(\jmath_x,\jmath_{x+1})]^\eta} \geq \delta$. That is, with $x \geq x_1$,

$$x[\varrho(\jmath_x, \jmath_{x+1})]^\eta \leq \rho x[\kappa(\varrho(\jmath_x, \jmath_{x+1})) - 1].$$

Hence, coming from Cases 1 and 2, we can find $\rho > 0$ and $x_1 \in \mathbb{N}$ such that all $x \geq x_1$,

$$x[\varrho(\jmath_x, \jmath_{x+1})]^\eta \leq \rho x[\kappa(\varrho(\jmath_x, \jmath_{x+1})) - 1].$$

By (12), we have, with $x \geq x_2 = \max\{x_0, x_1\}$,

$$x[\varrho(\jmath_x, \jmath_{x+1})]^\eta \leq \rho x[[\kappa(\varrho(\jmath_0, \jmath_1))]^{\left(\frac{b}{s}\right)^{(x-x_0)}} - 1]. \tag{15}$$

As $x \longrightarrow \infty$ in (15), we have $\lim\limits_{x\longrightarrow\infty} x[\varrho(\jmath_x, \jmath_{x+1})]^\eta = 0$.

Hence, we can find $x_3 \in \mathbb{N}$ such that $x[\varrho(\jmath_x, \jmath_{x+1})]^\eta \leq 1$ with $x \geq x_3$, which implies that

$$\varrho(\jmath_x, \jmath_{x+1}) \leq \frac{1}{x^{\frac{1}{\eta}}}. \tag{16}$$

Now, let $y, x \in \mathbb{N}$ with $y > x \geq x_3$. Then, by triangle inequality in $\aleph$, it follows from (16) that

$$\begin{aligned} \varrho(\jmath_x, \jmath_y) \leq\ & \varrho(\jmath_x, \jmath_{x+1}) + \varrho(\jmath_{x+1}, \jmath_{x+2}) + \cdots + \varrho(\jmath_{y-1}, \jmath_y) \\ =\ & \sum_{i=x}^{y-1} \varrho(\jmath_i, \jmath_{i+1}) \leq \sum_{i=x}^{\infty} \varrho(\jmath_i, \jmath_{i+1}) \\ \leq\ & \sum_{i=x}^{\infty} \frac{1}{i^{\frac{1}{\eta}}}. \end{aligned} \tag{17}$$

Since $\eta \in (0, 1)$, the series $\sum\limits_{i=x}^{\infty} \frac{1}{i^{\frac{1}{\eta}}}$ is convergent. Hence, for limit as $x \longrightarrow \infty$ in (17), gives $\varrho(\jmath_x, \jmath_y) \longrightarrow 0$. This shows that $\{\jmath_\flat\}_{\flat\in\mathbb{N}}$ is a Cauchy sequence in $(\aleph, \varrho)$. Hence, we can find $u \in \aleph$ such that $\jmath_x \longrightarrow u$ as $x \longrightarrow \infty$. To see that u is an intuitionistic fuzzy fixed point of Θ, assume that $u \notin [\Theta u]_{(\pi,\varsigma)}$ with $(\pi, \varsigma) \in (0, 1] \times [0, 1)$ and $\varrho(u, [\Theta u]_{(\pi,\varsigma)}) > 0$. Since $\varrho(\jmath_x, [\Theta \jmath_x]_{(\pi,\varsigma)}) \longrightarrow 0$ as $x \longrightarrow \infty$ and the function $\jmath \longmapsto \varrho(\jmath, [\Theta \jmath]_{(\pi,\varsigma)})$ is lsc, we obtain

$$0 \leq \varrho(u, [\Theta u]_{(\pi,\varsigma)}) \leq \liminf_{x\longrightarrow\infty} \varrho(\jmath_x, [\Theta \jmath_x]_{(\pi,\varsigma)}) = 0,$$

a contradiction. Consequently, we can find $u \in (0, 1]$ such that $u \in [\Theta u]_{(\pi,\varsigma)}$. $\square$

Remark 3. *If we consider $\mathcal{K}(\aleph)$ instead of $C_B(\aleph)$ in Theorem 8, we can remove the assumption (κ_4) on κ. Additionally, by considering Case 1 of Remark 2, we can let $s = 1$ and easily obtain the next result.*

Theorem 9. *Let $(\aleph, \varrho)$ be a complete metric-like space and $\Theta : \aleph \longrightarrow (IFset)^{\mathcal{K}(\aleph)}$ be an intuitionistic Fset-valued mapping. Suppose that Θ is a nonlinear intuitionistic Fset-valued κ-contraction of type(B), then Θ has an intuitionistic fuzzy fixed point in $\aleph$ on the criterion that $\jmath \longmapsto \varrho(\jmath, [\Theta \jmath]_{(\pi,\varsigma)})$ is lsc.*

Proof. Assume that Θ has no intuitionistic fuzzy fixed point in $\aleph$. Then, with $\jmath \in \aleph$ and $(\pi, \varsigma) \in (0, 1] \times [0, 1)$, $\varrho(\jmath, [\Theta \jmath]_{(\pi,\varsigma)}) > 0$. Since $[\Theta \jmath]_{(\pi,\varsigma)} \in \mathcal{K}(\aleph)$ for every $\jmath \in \aleph$, then by Case 1 of Remark 2, the set $\kappa_s^\jmath$ is nonempty. Hence, we can find $\ell \in \kappa_s^\jmath$ such that $\varrho(\jmath, \ell) = \varrho(\jmath, [\Theta \jmath]_{(\pi,\varsigma)})$. Let $\jmath_0 \in \aleph$ be an initial point. Then, from (7) and following the proof of Theorem 8, we have a Cauchy sequence $\{\jmath_\flat\}_{\flat\in\mathbb{N}}$ in $\aleph$ with $\jmath_{x+1} \in [\Theta \jmath_x]_{(\pi,\varsigma)}$, $\jmath_x \neq \jmath_{x+1}$

such that $\varrho(\jmath_x, \jmath_{x+1}) = \varrho(\jmath_x, [\Theta\jmath_x]_{(\pi,\varsigma)})$, $\kappa(\varrho(\jmath_{x+1}, [\Theta\jmath_{x+1}]_{(\pi,\varsigma)})) \leq [\kappa(\varrho(\jmath_x, \jmath_{x+1}))]^{\xi(\varrho(\jmath_x,\jmath_{x+1}))}$ and $\jmath_x \longrightarrow u$ as $x \longrightarrow \infty$. Since $\jmath \longmapsto \varrho(\jmath, [\Theta\jmath]_{(\pi,\varsigma)})$ is lsc, we obtain

$$0 \leq \varrho(u, [\Theta u]_{(\pi,\varsigma)}) \leq \liminf_{x \longrightarrow \infty} \varrho(\jmath_x, [\Theta\jmath_x]_{(\pi,\varsigma)}) = 0,$$

a contradiction. Hence, Θ has an intuitionistic fuzzy fixed point in $\aleph$. □

The next lemma will be employed in presenting our subsequent result.

Lemma 1. *Let $(\aleph, \varrho)$ be a metric-like space and $\Theta : \aleph \longrightarrow (IFset)^{\aleph}$ be an intuitionistic Fset-valued mapping such that $\hat{\Theta}\jmath \in \mathcal{C}_{\mathcal{B}}(\aleph)$ for each $\jmath \in \aleph$. Then $u \in \hat{\Theta}u$ if and only if*

$$\mu_{\Theta u}(u) \geq \mu_{\Theta u}(\jmath) \text{ and } \nu_{\Theta u}(u) \leq \nu_{\Theta u}(\jmath)\ \forall\ \jmath \in \aleph,$$

where $\hat{\Theta} : \aleph \longrightarrow \mathcal{C}_{\mathcal{B}}(\aleph)$ is a mapping induced by an intuitionistic Fset-valued mapping Θ, given as

$$\hat{\Theta}\jmath(t) = \{\ell \in \aleph : \mu_{\Theta_\jmath}(\ell) = \max_{t \in \aleph} \mu_{\Theta_\jmath}(t) \text{ and } \nu_{\Theta_\jmath}(\ell) = \min_{t \in \aleph} \nu_{\Theta_\jmath}(t)\}.$$

Proof. The proof is similar to the ideas of Azam and Tabassum ([19], Lemma 3.10). □

In the next result, we utilize Theorem 8 to deduce the existence of fixed point of intuitionistic Fset-valued mappings such that the contraction criterion does not explicitly involve a level-set.

Theorem 10. *Let $(\aleph, \varrho)$ be a complete metric-like space and $\Theta : \aleph \longrightarrow (IFset)^{\aleph}$ be an intuitionistic Fset-valued mapping. Assume further that the following conditions hold:*

(C1) $\hat{\Theta}\jmath \in \mathcal{C}_{\mathcal{B}}(\aleph)$ for each $\jmath \in \aleph$;

(C2) we can find $s \in (0,1)$ and a function $\xi : \mathbb{R}_+ \longrightarrow [0, s)$ satisfying

$$\limsup_{t \longrightarrow \varsigma^+} \xi(t) < s\ \forall \varsigma \in \mathbb{R}_+;$$

(C3) for each $\jmath \in \aleph$ with $\varrho(\jmath, \hat{\Theta}\jmath) > 0$, we can find $\ell \in \kappa_s^\jmath$ such that

$$\kappa(\varrho(\ell, \hat{\Theta}\ell)) \leq [\kappa(\varrho(\jmath, \ell))]^{\xi(\varrho(\jmath,\ell))},$$

where $\kappa \in \mho$;

(C4) the function $\jmath \longmapsto \varrho(\jmath, \hat{\Theta}\jmath)$ is lsc.

Then, we can find $u \in \aleph$ such that for all $\jmath \in \aleph$,

$$\mu_{\Theta u}(u) \geq \mu_{\Theta u}(\jmath) \text{ and } \nu_{\Theta u}(u) \leq \nu_{\Theta u}(\jmath).$$

Proof. Suppose that

$$\max_{t \in \aleph} \mu_{\Theta_\jmath}(t) = \varrho_1 \text{ and } \min_{t \in \aleph} \nu_{\Theta_\jmath} = \varrho_2.$$

Then, $\hat{\Theta}\jmath = [\Theta\jmath]_{(\varrho_1,\varrho_2)}$ for each $\jmath \in \aleph$. Consequently,

$$\begin{aligned} \kappa(\varrho(\ell, [\Theta\ell]_{(\varrho_1,\varrho_2)})) &= \kappa(\varrho(\ell, \hat{\Theta}\ell)) \\ &\leq [\kappa(\varrho(\jmath, \ell))]^{\xi(\varrho(\jmath,\ell))}. \end{aligned}$$

Hence, by Theorem 8, we can find $u \in \aleph$ such that $u \in [\Theta u]_{(\varrho_1,\varrho_2)} = \hat{\Theta}u$. By Lemma 1,

$$\mu_{\Theta u}(u) \geq \mu_{\Theta u}(\jmath) \text{ and } \nu_{\Theta u}(u) \leq \nu_{\Theta u}(\jmath)\ \forall \jmath \in \aleph.$$

□

In the following, we develop an illustration to support the hypotheses of Theorems 8 and 9.

Example 2. *Let $\aleph = [0,2] \cup \left\{-\frac{1}{x} : x \geq 1\right\}$ and define the mapping $\varrho : \aleph \times \aleph \longrightarrow$ by*

$$\varrho(\jmath, \ell) = \begin{cases} \max\{|\jmath|, |\ell|\}, & \textit{if } \jmath, \ell \in [0,2] \\ |\jmath| + |\ell|, & \textit{otherwise.} \end{cases}$$

Then $(\aleph, \varrho)$ is a complete metric-like space. Note that ϱ is not a metric on $\aleph$, since $\varrho(1,1) = 1$. Let $\lambda, \zeta \in (0,1]$ and define an intuitionistic Fset-valued mapping $\Theta = \langle \mu_\Theta, \nu_\Theta \rangle : \aleph \longrightarrow (IFset)^{\aleph}$ as follows:

Case 1. For $\jmath \in [0,2]$,

$$\mu_{\Theta(\jmath)}(t) = \begin{cases} \frac{\lambda}{2}, & \textit{if } t = \jmath \\ \frac{\lambda}{4}, & \textit{if } t = \frac{\jmath+2}{\jmath+3} \\ \frac{\lambda}{30}, & \textit{if } t = \frac{\jmath+3}{\jmath+7} \\ 0, & \textit{otherwise,} \end{cases} \quad \nu\Theta(\jmath)(t) = \begin{cases} 0, & \textit{if } t = \jmath \\ \frac{\zeta}{16}, & \textit{if } t = \frac{\jmath+2}{\jmath+3} \\ \frac{\zeta}{9}, & \textit{if } t = \frac{\jmath+3}{\jmath+7} \\ \zeta, & \textit{otherwise.} \end{cases}$$

Case 2. For $\jmath \notin [0,2]$,

$$\mu_{\Theta(\jmath)}(t) = \begin{cases} \lambda, & \textit{if } t = \jmath - \frac{1}{2} \\ \frac{\lambda}{3}, & \textit{if } t = \jmath - \frac{1}{4} \\ \frac{\lambda}{4}, & \textit{if } t = \jmath - \frac{1}{8} \\ 0, & \textit{otherwise,} \end{cases} \quad \nu\Theta(\jmath)(t) = \begin{cases} 0, & \textit{if } t = \jmath - \frac{1}{2} \\ \frac{\zeta}{70}, & \textit{if } t = \jmath - \frac{1}{4} \\ \frac{\zeta}{16}, & \textit{if } t = \jmath - \frac{1}{8} \\ \zeta, & \textit{otherwise.} \end{cases}$$

Take $(\pi, \varsigma) = \left(\frac{\lambda}{4}, \frac{\zeta}{16}\right)$, then from Cases (i) and (ii), we have

$$[\Theta\jmath]_{(\pi,\varsigma)} = \begin{cases} \left\{\jmath, \frac{\jmath+2}{\jmath+3}\right\}, & \textit{if } 0 \leq \jmath \leq 2 \\ \left\{\jmath - \frac{1}{2}, \jmath - \frac{1}{4}, \jmath - \frac{1}{8}\right\}, & \textit{if } \jmath = -\frac{1}{x},\ x \geq 1. \end{cases}$$

Very apparent, $[\Theta\jmath]_{(\pi,\varsigma)} \in \mathcal{K}(\aleph) \subset \mathcal{C}_{\mathcal{B}}(\aleph)$. By direct calculation, we see that

$$\varrho(\jmath, [\Theta\jmath]_{(\pi,\varsigma)}) = \begin{cases} \jmath, & \textit{if } \jmath \in [0,2] \\ 2\left(\jmath - \frac{1}{16}\right), & \textit{if } \jmath \notin [0,2], \end{cases}$$

and hence the function $\jmath \longmapsto \varrho(\jmath, \Theta\jmath]_{(\pi,\varsigma)})$ is lsc.

Next, we will show that the contraction inequality (5) holds. Let $\kappa(t) = e^{\sqrt{te^t}}$, then (5) is converted to

$$\frac{\varrho(\ell, [\Theta\ell]_{(\pi,\varsigma)})}{\varrho(\jmath, \ell)} e^{\varrho(\ell, [\Theta\ell]_{(\pi,\varsigma)}) - \varrho(\jmath,\ell)} \leq [\xi(\varrho(\jmath, \ell))]^2. \tag{18}$$

So, we will check alternatively that Θ satisfies (18). Take $s \in \left(\sqrt{e^{-\frac{5}{8}}}, 1\right)$ and set up $\xi : \mathbb{R}_+ \longrightarrow [0, s)$ as

$$\xi(t) = \begin{cases} \sqrt{e^{-\left[\frac{(t+3)^2-3}{t+3}\right]}}, & \textit{if } 0 \leq t \leq 2 \\ \sqrt{e^{-\frac{5}{8}}}, & \textit{otherwise.} \end{cases}$$

If $\varrho(\jmath, \Theta\jmath]_{(\pi,\varsigma)}) > 0$, *then* $\jmath \neq 0$. *Hence, for* $\jmath \in (0,2]$, *we have* $\ell = \frac{\jmath+2}{\jmath+3} \in \kappa_s^\jmath$ *such that*

$$\begin{aligned} \frac{\varrho(\ell, [\Theta\ell]_{(\pi,\varsigma)})}{\varrho(\jmath,\ell)} e^{\varrho(\ell,[\Theta\ell]_{(\pi,\varsigma)})-\varrho(\jmath,\ell)} &= \frac{\jmath+2}{\jmath(\jmath+3)} e^{\frac{\jmath+2}{\jmath+3}-\jmath} \\ &\leq e^{-\left(\frac{\jmath^2+2\jmath-2}{\jmath+3}\right)} = e^{-\left[\frac{(\jmath+1)^2-3}{\jmath+3}\right]} \\ &= [\xi(\jmath)]^2 = [\xi(\varrho(\jmath,\ell))]^2. \end{aligned}$$

Furthermore, for $\jmath \notin [0,2]$, *we have* $\ell = \jmath - \frac{1}{2} \in \kappa_s^\jmath$ *with* $s \in \left(\sqrt{e^{-\frac{5}{8}}}, 1\right)$ *such that*

$$\begin{aligned} \frac{\varrho(\ell, [\Theta\ell]_{(\pi,\varsigma)})}{\varrho(\jmath,\ell)} e^{\varrho(\ell,[\Theta\ell]_{(\pi,\varsigma)})-\varrho(\jmath,\ell)} &= \left(\frac{\jmath-\frac{9}{16}}{\jmath-\frac{1}{4}}\right) e^{2[(\jmath-\frac{9}{16})-(\jmath-\frac{1}{4})]} \\ &\leq e^{-\frac{5}{8}} = \left[\xi\left(2\jmath-\frac{1}{8}\right)\right]^2 \\ &= [\varrho(\jmath,\ell)]^2. \end{aligned}$$

Hence, all the hypotheses of Theorems 8 and 9 are obeyed. Consequently, we can see that Θ *has an intuitionistic fuzzy fixed point in* $\aleph$.

4. Consequences

We highlight some special results of Theorems 8 and 9 in this section.

Corollary 1. *Let* $(\aleph, \varrho)$ *be a complete metric-like space,* $\Theta : \aleph \longrightarrow (IFset)^{\mathcal{C}(\aleph)}$ *be an intuitionistic Fset-valued mapping and* $\kappa \in \mho$. *Suppose that we can find* $\xi, s \in (0,1)$ *with* $\xi < s$ *and* $\ell \in \kappa_s^\jmath$ *such that*

$$\kappa(\varrho(\ell, [\Theta\ell]_{(\pi,\varsigma)})) \leq [\kappa(\varrho(\jmath, [\Theta\jmath]_{(\pi,\varsigma)}))]^\xi,$$

for each $\jmath \in \aleph$ *with* $\varrho(\jmath, [\Theta\jmath]_{(\pi,\varsigma)}) > 0$, *then* Θ *has an intuitionistic fuzzy fixed point in* $\aleph$ *on the criterion that* $\jmath \longmapsto \varrho(\jmath, [\Theta\jmath]_{(\pi,\varsigma)})$ *is lsc.*

Corollary 2. *Let* $(\aleph, \varrho)$ *be a complete metric-like space,* $\Theta : \aleph \longrightarrow (IFset)^{\mathcal{K}(\aleph)}$ *be an intuitionistic Fset-valued mapping and* $\kappa \in \Omega$. *Suppose that we can find* $\xi \in (0,1)$ *and* $\ell \in \kappa_s^\jmath$ *such that*

$$\kappa(\varrho(\ell, [\Theta\ell]_{(\pi,\varsigma)})) \leq [\kappa(\varrho(\jmath, [\Theta\jmath]_{(\pi,\varsigma)}))]^\xi,$$

for each $\jmath \in \aleph$ *with* $\varrho(\jmath, [\Theta\jmath]_{(\pi,\varsigma)}) > 0$, *then* Θ *has an intuitionistic fuzzy fixed point in* $\aleph$ *on the criterion that* $\jmath \longmapsto \varrho(\jmath, [\Theta\jmath]_{(\pi,\varsigma)})$ *is lsc.*

Since $\kappa_s^\jmath \subset [\Theta\jmath]_{(\pi,\varsigma)}$, more ideas can be discovered as follows.

Corollary 3. *Let* $(\aleph, \varrho)$ *be a complete metric-like space,* $\Theta : \aleph \longrightarrow (IFset)^{\mathcal{C}(\aleph)}$ *be an intuitionistic Fset-valued mapping. Suppose that we can find* $\xi \in (0,1)$ *and* $\ell \in [\Theta\jmath]_{(\pi,\varsigma)}$ *such that*

$$\kappa(\varrho(\ell, [\Theta\ell]_{(\pi,\varsigma)})) \leq [\kappa(\varrho(\jmath, [\Theta\jmath]_{(\pi,\varsigma)}))]^\xi,$$

for each $\jmath \in \aleph$ *with* $\varrho(\jmath, [\Theta\jmath]_{(\pi,\varsigma)}) > 0$, *then* Θ *has an intuitionistic fuzzy fixed point in* $\aleph$ *on the criterion that* $\jmath \longmapsto \varrho(\jmath, [\Theta\jmath]_{(\pi,\varsigma)})$ *is lsc.*

Employing the ideas of Feng and Liu [29], for $a \in (0,1)$ and all $\jmath \in \aleph$, set up the set $I_a^\jmath \subset \aleph$ as:

$$I_a^\jmath = \{\ell \in [\Theta\jmath]_{(\pi,\varsigma)} \,|\, a\varrho(\jmath,\ell) \leq \varrho(\jmath, [\Theta\jmath]_{(\pi,\varsigma)}) \text{ for each } (\pi,\varsigma) \in (0,1] \times [0,1)\}.$$

Corollary 4. *Let $(\aleph, \varrho)$ be a complete metric-like space, $\Theta : \aleph \longrightarrow (IFset)^{\mathcal{C}(\aleph)}$ be an intuitionistic Fset-valued mapping. Suppose that we can find $c \in (0,1)$ such that each $\jmath \in \aleph$, there is $\ell \in I_a^\jmath$ satisfying*

$$\varrho(\ell, [\Theta\ell]_{(\pi,\varsigma)}) \leq c\varrho(\jmath, \ell),$$

then Θ has an intuitionistic fuzzy fixed point in $\aleph$ on the criterion that $\jmath \longmapsto \varrho(\jmath, [\Theta\jmath]_{(\pi,\varsigma)})$ is lsc.

Proof. Define $\kappa(t) = e^{\sqrt{t}}$, $\xi = \sqrt{c}$ and $s = \sqrt{a}$ in the special result Corollary 1. □

5. Application in Ordered Metric-like Spaces

The concept of fixed point on metric space with a partial order is attracting increasing attention in the field of fixed point theory. This progress was brought about by Turinici [36] in 1986, but subsequently became one of the deeply researched projects following the announcement of the work of Ran and Reurings [37] and Nieto and Rodriguez [38].

At this junction, we study the version of our earlier results in metric-like space provided with a partial order. Accordingly, $(\aleph, \varrho, \preceq)$ is called an ordered metric-like space if:

(i) $(\aleph, \varrho)$ is a metric-like space, and
(ii) $(\aleph, \preceq)$ is a partially ordered set.

Any two elements $\jmath, \ell \in \aleph$ are said to be comparable if either $\jmath \preceq \ell$ or $\ell \preceq \jmath$ holds. Let $\Theta : \aleph \longrightarrow (IFset)^{\aleph}$ be an intuitionistic Fset-valued mapping. For each $\jmath \in \aleph$ with $\varrho(\jmath, [\Theta\jmath]_{(\pi,\varsigma)}) > 0$ and with $(\pi, \varsigma) \in (0,1] \times [0,1)$, set up the set $\kappa_s^{\jmath,\preceq} \subseteq \aleph$, $s \in (0,1]$ as

$$\kappa_s^{\jmath,\preceq} = \{\ell \in [\Theta\jmath]_{(\pi,\varsigma)} : [\kappa(\varrho(\jmath,\ell))]^s \leq \kappa(\varrho(\jmath,\ell)),\ \jmath \preceq \ell\}.$$

Definition 10. *Let $\aleph$ be a nonempty set. An intuitionistic Fset-valued mapping $\Theta : \aleph \longrightarrow (IFset)^{\aleph}$ is (π,ς)-comparative if we can find $(\pi,\varsigma) \in (0,1] \times [0,1)$ such that for each $\jmath \in \aleph$ and $\ell \in [\Theta\jmath]_{(\pi,\varsigma)}$ with $\jmath \preceq \ell$, we have $\ell \preceq u$ and $u \in [\Theta\ell]_{(\pi,\varsigma)}$.*

Theorem 11. *Let $(\aleph, \varrho, \preceq)$ be a complete metric-like space, $\Theta : \aleph \longrightarrow (IFset)^{\mathcal{C}(\aleph)}$ be an intuitionistic Fset-valued mapping and $\kappa \in \mho$. Assume that*

(C1) *the mapping $\jmath \longmapsto \varrho(\jmath, [\Theta\jmath]_{(\pi,\varsigma)})$ is ordered lsc;*
(C2) *we can find a constant $s \in (0,1)$ and a mapping $\xi : \mathbb{R}_+ \longrightarrow [0,s)$ such that*

$$\limsup_{t \longrightarrow \varsigma^+} \xi(t) < s \text{ for all } \varsigma \in \mathbb{R}_+;$$

(C3) *for each $\jmath \in \aleph$, we can find $\ell \in \kappa_s^{\jmath,\preceq}$ with $\jmath \preceq \ell$ such that*

$$\kappa(\varrho(\jmath, [\Theta\ell]_{(\pi,\varsigma)})) \leq [\kappa(\varrho(\jmath,\ell))]^{\xi(\varrho(\jmath,\ell))};$$

(C4) *Θ is (π,ς)-comparative;*
(C5) *if $\{\jmath_\flat\}_{\flat \in \mathbb{N}} \subset \aleph$ with $\jmath_{x+1} \in [\Theta\jmath_x]_{(\pi,\varsigma)}$, $\jmath_x \longrightarrow u \in \aleph$ as $x \longrightarrow \infty$, then $\jmath_x \preceq u$ with $x \in \mathbb{N}$.*

Proof. In line with the proof of Theorem 8 and the fact that $\kappa_s^{\jmath,\preceq} \subseteq \aleph$, we can show that $\{\jmath_\flat\}_{\flat \in \mathbb{N}}$ is a Cauchy sequence in $(\aleph, \varrho, \preceq)$ with $\jmath_{x-1} \preceq \jmath_x, \in \mathbb{N}$. The completeness of this space implies that we can find $u \in \aleph$ such that $\jmath \longrightarrow u$ as $x \longrightarrow \infty$. By Condition (C5), $\jmath_x \preceq u$ with $x \in \mathbb{N}$. From this point, Theorem 8 can be applied to find $u \in \aleph$ such that $u \in [\Theta u]_{(\pi,\varsigma)}$. □

6. Application in Metric-like Spaces Furnished with a Binary Relation

Let $(\aleph, \varrho, \mathcal{R})$ be a binary metric-like space, where $\mathcal{R}$ is a binary relation on $\aleph$. Define $\mathfrak{B} = \mathcal{R} \cup \mathcal{R}^{-1}$. It is easy to notice that with $\jmath, \ell \in \aleph$, $\jmath\mathfrak{B}\ell$ if and only if $\jmath\mathcal{R}\ell$ or $\ell\mathcal{R}\jmath$.

Definition 11. *Let $\aleph$ be a nonempty set. We say that an intuitionistic Fset-valued mapping $\Theta : \aleph \longrightarrow (IFset)^{\aleph}$ is $(\pi, \varsigma, \mathcal{R})$-comparative, if we can find $(\pi, \varsigma) \in (0,1] \times [0,1)$ and a binary relation $\mathcal{R}$ on $\aleph$ such that each $\jmath \in \aleph$ and $\ell \in [\Theta\jmath]_{(\pi,\varsigma)}$ with $\jmath\mathfrak{B}\ell$, we have $\ell\mathfrak{B}u$ and $u \in [\Theta\ell]_{(\pi,\varsigma)}$.*

Definition 12. *Let $(\aleph, \varrho, \mathcal{R})$ be a metric-like space furnished with a binary relation $\mathcal{R}$ and $\Theta : \aleph \longrightarrow (IFset)^{\aleph}$ be an intuitionistic Fset-valued mapping. A function $g : (\aleph, \varrho, \mathcal{R}) \longrightarrow \mathbb{R}$ is called binary lsc, if $g(u) \leq \liminf_{x \longrightarrow \infty} g(\jmath_x)$ for every sequence $\{\jmath_x\}_{\mathbb{N}}$ in $\aleph$ with $[\Theta\jmath_x]_{(\pi,\varsigma)}\mathfrak{B}[\Theta\jmath_{x+1}]_{(\pi,\varsigma)}$, $x \in \mathbb{N}$ and $\jmath_x \longrightarrow u \in \aleph$ as $x \longrightarrow \infty$.*

For each $\jmath \in \aleph$ with $\varrho(\jmath, [\Theta\jmath]_{(\pi,\varsigma)}) > 0$, and a binary relation $\mathcal{R}$ on $\aleph$, set up the set $\kappa_s^{\jmath,\mathcal{R}}$, $s \in (0,1]$ as:

$$\kappa_s^{\jmath,\mathcal{R}} = \{\ell \in [\Theta\jmath]_{(\pi,\varsigma)} : [\kappa(\varrho(\jmath,\ell))]^s \leq \kappa(\varrho(\jmath,\ell)),\ \jmath\mathfrak{B}\ell\}.$$

Theorem 12. *Let $(\aleph, \varrho, \mathcal{R})$ be a binary complete metric-like space, $\Theta : \aleph \longrightarrow (IFset)^{\mathcal{C}(\aleph)}$ be an intuitionistic Fset-valued mapping and $\kappa \in \mho$. Assume that*
$(C1)$ *the mapping $\jmath \longmapsto \varrho(\jmath, [\Theta\jmath]_{(\pi,\varsigma)})$ is binary lsc;*
$(C2)$ *we can find $s \in (0,1)$ and a function $\xi : \mathbb{R}_+ \longrightarrow [0,s)$ satisfying*

$$\limsup_{t \longrightarrow \varsigma^+} \xi(t) < s \text{ for all } \varsigma \in \mathbb{R}_+,$$

$(C3)$ *for each $\jmath \in \aleph$, we can find $\ell \in \kappa_s^{\jmath,\mathcal{R}}$ with $\jmath\mathfrak{B}\ell$ such that*

$$\kappa(\varrho(\jmath, [\Theta\ell]_{(\pi,\varsigma)})) \leq [\kappa(\varrho(\jmath,\ell))]^{\xi(\varrho(\jmath,\ell))};$$

$(C4)$ *Θ is $(\pi, \varsigma, \mathcal{R})$-comparative;*
$(C5)$ *if $\{\jmath_\flat\}_{\flat\in\mathbb{N}} \subset \aleph$ with $\jmath_{x+1} \in [\Theta\jmath_x]_{(\pi,\varsigma)}$, $\jmath_x \longrightarrow u \in \aleph$ as $x \longrightarrow \infty$, then $\jmath_x\mathfrak{B}u$, $x \in \mathbb{N}$.*

Proof. The proof follows similar ideas of Theorem 11. □

7. Application in Multivalued Mappings in Metric-like Spaces

In this section, we utilize some results from the previous section to deduce their classical multivalued similitudes in the framework of metric-like spaces. It is well-known that metric-like spaces cannot be Hausdorff(for details, see [35]), making it impossible for the usual studies of fixed point of set-valued mappings via the Hausdorff metric. Though, using the Feng and Liu's technique (see [29]), this shortcoming can be overcome. Hence, as far as we know, all the results presented herein are also novel as are refinements of their counterparts in metric space.

Let $F : \aleph \longrightarrow \mathcal{N}(\aleph)$ be a multivalued mapping. Denoted by $\Lambda_s^{\jmath}$, the set

$$\Lambda_s^{\jmath} = \{\ell \in F\jmath : [\kappa(\varrho(\jmath,\ell))]^s \leq \kappa(\varrho(\jmath, F\jmath))\}.$$

Theorem 13. *Let $(\aleph, \varrho)$ be a complete metric-like space, $\Im : \aleph \longrightarrow \mathcal{C}(\aleph)$ be a multivalued mapping and $\kappa \in \mho$. Suppose that we can find $s \in (0,1)$ and a function $\xi : \mathbb{R}_+ \longrightarrow [0,s)$ satisfying*

$$\limsup_{t \longrightarrow \varsigma^+} \xi(t) < s \text{ for all } \varsigma \in \mathbb{R}_+,$$

and for each $\jmath \in \aleph$ with $\varrho(\jmath, F\jmath) > 0$, there is $\ell \in \Lambda_s^{\jmath}$ such that

$$\kappa(\varrho(\ell, F\ell)) \leq [\kappa(\varrho(\jmath,\ell))]^{\xi(\varrho(\jmath,\ell))}, \tag{19}$$

then F has a fixed point in $\aleph$ on the criterion that $\jmath \longmapsto \varrho(\jmath, F\jmath)$ is lsc.

Proof. Consider an intuitionistic Fset-valued mapping $\Theta = \langle \mu_\Theta, \nu_\Theta \rangle : \aleph \longrightarrow (IFset)^{\aleph}$ set up by

$$\mu_{\Theta(\jmath)}(t) = \begin{cases} 1, & \text{if } t \in F\jmath \\ 0, & \text{if } t \notin F\jmath, \end{cases} \quad \nu_{\Theta(\jmath)}(t) = \begin{cases} 0, & \text{if } t \in F\jmath \\ 1, & \text{if } t \notin F\jmath. \end{cases}$$

Then, we can find $(\pi, \varsigma) = (1,0) \in (0,1] \times [0,1)$ such that $[\Theta\jmath]_{(1,0)} = F\jmath$. Hence, Theorem 8 can be applied to find $u \in \aleph$ such that $u \in [\Theta u]_{(1,0)} = Fu$. □

On the same line of deducing Theorem 13, the next two results can be obtained from Theorem 9 and Corollary 4, individually.

Theorem 14. *Let $(\aleph, \varrho)$ be a complete metric-like space, $F : \aleph \longrightarrow \mathcal{K}(\aleph)$ be a multivalued mapping and $\kappa \in \Omega$. Suppose that we can find a function $\xi : \mathbb{R}_+ \longrightarrow [0,1)$ such that*

$$\limsup_{t \longrightarrow \varsigma^+} \xi(t) < 1 \text{ for all } \varsigma \in \mathbb{R}_+,$$

and for each $\jmath \in \aleph$ with $\varrho(\jmath, F\jmath) > 0$, there is $\ell \in \Lambda_1^{\jmath}$ such that

$$\kappa(\varrho(\ell, Fy)) \leq [\kappa(\varrho(\jmath, \ell))]^{\xi(\varrho(\jmath,\ell))},$$

then F has a fixed point in $\aleph$ on the criterion that $\jmath \longmapsto \varrho(\jmath, F\jmath)$ is lsc.

Theorem 15. *Let $(\aleph, \varrho)$ be a complete metric-like space, $\Im : \aleph \longrightarrow \mathcal{C}(\aleph)$ be a multivalued mapping. Suppose that we can find $a, c \in (0,1)$ such that each $\jmath \in \aleph$, there is $\ell \in I_a^{\jmath}$ such that*

$$\varrho(\ell, F\ell) \leq c\varrho(\jmath, \ell),$$

then F has a fixed point in $\aleph$ on the criterion that $c < a$ and the function $\jmath \longmapsto \varrho(\jmath, F\jmath)$ is lsc.

Remark 4.

(i) *Theorems 8 and 9 are intuitionistic fuzzy refinements of the results of Altun and Minak [31] as well as Durmaz and Altun [33] in the framework of metric-like spaces.*

(ii) *Theorems 13 and 14 are extensions of the results in [33] and Theorems 3 and 4 due to Klim and Wardowski [30] in the setting of metric-like spaces.*

(iii) *Theorem 15 is a proper refinement of Theorem 2 due to Feng and Liu [29].*

(iv) *It is obvious that Theorems 8–15 can all be reduced to their Fset equivalence.*

8. Conclusions

In this paper, the notion of nonlinear intuitionistic Fset-valued κ-contractions in the framework of metric-like spaces has been launched. Subsequently, by using the approach of Feng and Liu [29] together with the idea of κ-contraction due to Jleli and Samet [28], the existence of intuitionistic fuzzy fixed point for the new contractions are proved. As some utilizations of the principal theorems, a few fixed point results of metric-like spaces equipped with partial ordering and binary relations as well as classical set-valued mappings have been derived. The concepts examined herein generalize, complement and unify a number of significant results in the corresponding literature; some of these special cases were highlighted in [12,29–31]. It is pertinent to highlight that the ideas presented in this paper can be re-examined in several other ways, viz., the metric-like component can be extended to some generalized dislocated metric spaces. The intuitionistic fuzzy mappings can be considered in terms of L-fuzzy mappings, and some hybrid nonclassical mappings.

While the idea proposed in this paper is theoretical, it can however help us to comprehend a number of useful problems in high particle physics, since the events in this realm are mostly intutitionistic fuzzy in nature (see, e.g., [39]). It is also well-known that there are enormous applications of fuzziness in physics, a prototypal limit through which we can understand the behaviours of quantum particles and the negligible scales of nature.

These scales help us to measure the possibilities concerning the existence and behaviours of substances. In these directions, our results herein may yield an amount of fuzziness needed in quantum theory and the development of other scientific areas where there is information with nonstatistical uncertainties.

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