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Recurrent Sequences Play for Survival Probability of Discrete Time Risk Model

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Abstract: In this article we investigate a homogeneous discrete time risk model with a generalized premium income rate which can be any natural number. We derive theorems and give numerical examples for finite and ultimate time survival probability calculation for the mentioned model. Our proved statements for ultimate time survival probability calculation, at some level, are similar to the previously known statements for non-homogeneous risk models, where required initial values of survival probability for some recurrent formulas are gathered by certain limit laws. We also give a simplified proof that a ruin is almost unavoidable with a neutral net profit condition and state several conjectures on a certain type of recurrent matrices non-singularity. All the research done can be interpreted as a possibility that symmetric or asymmetric random walk (r.w.) hits (or not) the line $u+\kappa t$ and that possibility is directly related to the expected value of r.w. generating random variable which might be equal, above or bellow κ.

Keywords: discrete time risk model; random walk; ruin probability; survival probability; ultimate time; net profit condition

MSC: 91G05; 60G50

1. Introduction

A game of gain and loss occurs in various situations. All individuals have savings, earn income, and face expenses. Obviously, expenses, being greater than income and savings, cause inconveniences or bankruptcy. Models, trying to express such situations and measure its likelihood, are often random walk (sum of certain random variables) based. In general, random walk has various occasions: pure mathematics, insurance, engineering, computer science, physics, and many others—across all natural and related sciences. Our work is both pure and insurance mathematics shifted and we calculate probability that a certain increasing random amount will never hit some selected increasing threshold. The mentioned event is directly related to some equilibrium conditions which, in separate cases, might be deemed as axes of symmetry.

One of the most general risk models in collective risk theory is the Sparre Andersen's risk model presented in [1]. This model assumes that the insurers surplus process W has the following expression:

$$W(t) := u + ct - \sum_{i=1}^{\Theta(t)} Z_i,\ t \geqslant 0, \tag{1}$$

where:

- $u \geqslant 0$ denotes the initial insurer's surplus;
- $c > 0$ denotes the premium rate per unit of time;
- The cost of claims $\{Z_1, Z_2, \ldots\}$ are independent copies of a non-negative random variable (r.v.) Z;
- The inter-occurrence times of claims $\{\theta_1, \theta_2, \ldots\}$ are another sequence of independent copies of a non-negative r.v. θ, which is not degenerate at zero;
- The sequences $\{Z_1, Z_2, \ldots\}$ and $\{\theta_1, \theta_2, \ldots\}$ are mutually independent;
- $\Theta(t) = \#\{n \geqslant 1 : T_n \leqslant t\}$ is the renewal process generated by r.v. θ, where $T_n = \theta_1 + \theta_2 + \ldots + \theta_n$.

Since 1957, when the Sparre Andersen's risk model (1) was introduced, there occurred a significant amoutn of research papers across the world on certain versions of the model (1). For example, [2–12] and many others. An observable break in the subject was achieved when [7,13,14] were published in 1988.

In this paper, we consider the special case of the general Sparre Andersen's model. In (1), we set $c = \kappa \in \mathbb{N}, \theta \equiv 1$, and $Z_i \overset{d}{=} X_i$, $i \in \mathbb{N}$, where X_i are independent copies of an integer valued non-negative r.v. X. Under such restrictions, for the insurers surplus process W, we get the following expression:

$$W(t) = u + \kappa t - \sum_{i=1}^{\lfloor t \rfloor} X_i,\ t \geqslant 0. \tag{2}$$

Since a r.v. X is discrete, it is enough to consider $u \in \mathbb{N}_0 := \{0, 1, \ldots\}$ and $t \in \mathbb{N}$ for the defined model (2). Therefore, the model we work with is given by the following formula:

$$W(t) = u + \kappa t - \sum_{i=1}^{t} X_i,\ u \in \mathbb{N}_0,\ t,\ \kappa \in \mathbb{N},\ X_i \overset{d}{=} X \tag{3}$$

and we call it the generalized premium discrete time risk model (GPDTRM). In addition, it is natural to define that $W(0) := u$. The finite and ultimate time survival probabilities for the model presented by (3) are correspondingly defined as:

$$\varphi(u, T) := \mathbb{P}\left(\bigcap_{t=1}^{T} \{W(t) > 0\}\right),\ \varphi(u) := \mathbb{P}\left(\bigcap_{t=1}^{\infty} \{W(t) > 0\}\right), \tag{4}$$

where $T \in \mathbb{N}$.

In addition, for each $i \in \mathbb{N}$, let us denote the local probabilities of r.v. X by $h_i := \mathbb{P}(X = i)$, value of the accumulated distribution function of X by $H(i) := \mathbb{P}(X \leqslant i)$, and value of the tail distribution of X by $\overline{H}(i) := 1 - H(i) = \mathbb{P}(X > i)$.

The following simple statement provide us the algorithm to calculate values of the finite time survival probability.

Theorem 1. *For GPDTRM presented by Formula (3), the finite time ruin probability satisfies the following equations:*

$$\varphi(u, 1) = H(u + \kappa - 1),\ \ \varphi(u, T) = \sum_{i=0}^{u+\kappa-1} \varphi(u + \kappa - i, T - 1) h_i,$$

where $u \geqslant 0$ and $T \geqslant 2$.

Proof. For $T = 1$, the formula follows straightforward by the finite time survival probability definition (4): $\varphi(u,1) = \mathbb{P}(u + \kappa - X_1 > 0) = H(u + \kappa - 1)$, and for $T = 2, 3, \ldots$ it follows by the law of total probability and elementary rearrangements:

$$\varphi(u,T) = \mathbb{P}\left(\bigcap_{t=1}^{T}\left\{u + \kappa t - \sum_{i=1}^{t} X_i > 0\right\}\right) = \mathbb{P}\left(\bigcap_{t=2}^{T}\left\{u + \kappa t - \sum_{i=1}^{t} X_i > 0\right\}, X_1 \leqslant u + \kappa - 1\right)$$
$$= \sum_{i=0}^{u+\kappa-1} \mathbb{P}\left(\bigcap_{t=1}^{T-1}\left\{u + \kappa t + \kappa - i - \sum_{i=1}^{t} X_i > 0\right\}, X_1 = i\right) = \sum_{i=0}^{u+\kappa-1} \varphi(u + \kappa - i, T - 1)\, h_i.$$

□

Our reason to present Theorem 1 is to see the broader view calculating the ultimate time survival probability in Section 5 below. From definition (4) it is easy to see that $\varphi(u, T+1) \leqslant \varphi(u, T)$ and $\varphi(u, T) \leqslant \varphi(u+1, T)$ for all $u \in \mathbb{N}_0$ and $T \in \mathbb{N}$.

Let us turn to the ultimate time survival probability. By similar arguments as in proof of Theorem 1, the ultimate time survival probability of the model (3) for all $u \in \mathbb{N}_0$ satisfies the following relation:

$$\varphi(u) = \sum_{i=1}^{u+\kappa} h_{u+\kappa-i}\, \varphi(i). \tag{5}$$

Indeed, by the same arguments as in the proof of Theorem 1, we get:

$$\varphi(u) = \mathbb{P}\left(\bigcap_{t=1}^{\infty}\left\{u + \kappa t - \sum_{i=1}^{t} Z_i > 0\right\}\right)$$
$$= \mathbb{P}\left(\bigcap_{t=1}^{\infty}\left\{u + \kappa t - \sum_{i=1}^{t} Z_i > 0\right\}, Z_1 \leqslant u + \kappa - 1\right) + \mathbb{P}\left(\bigcap_{t=1}^{\infty}\left\{u + \kappa t - \sum_{i=1}^{t} Z_i > 0\right\}, Z_1 > u + \kappa - 1\right)$$
$$= \sum_{i=0}^{u+\kappa-1} \mathbb{P}\left(\bigcap_{t-1=1}^{\infty}\left\{u + \kappa(t-1) + \kappa - Z_1 - \sum_{i=1}^{t-1} Z_i > 0\right\}, Z_1 = i\right)$$
$$= \sum_{i=0}^{u+\kappa-1} \mathbb{P}\left(\bigcap_{t=1}^{\infty}\left\{u + \kappa t + \kappa - i - \sum_{i=1}^{t} Z_i > 0\right\}\right) h_i = \sum_{i=0}^{u+\kappa-1} \varphi(u + \kappa - i)\, h_i = \sum_{i=1}^{u+\kappa} h_{u+\kappa-i}\, \varphi(i).$$

We can see from the derived recurrence relation (5) that to get the value of $\varphi(u+k)$ we must know all the previous values $\varphi(0), \varphi(1), \ldots, \varphi(u+k-1)$ even in the case of $u = 0$. In fact, we further do spins around the finding of those initial values. But first, we need to describe a net profit condition.

It is said that the net profit condition for the GPDTRM (3) holds if:

$$\mathbb{E}X - \kappa < 0. \tag{6}$$

The intuitive explanation of this condition is simple. Let us rewrite the main model Equation (3) by the form:

$$W(t) = u + \sum_{i=1}^{t} (\kappa - X_i).$$

From this, it follows that $\mathbb{E}W(t) = u - t(\mathbb{E}X - \kappa)$ and only condition $\mathbb{E}W(t) > 0$ allows us to expect that $W(t) > 0$ with some non-zero probability for all $t \in \{1, 2, \ldots\}$. In Sections 2 and 3, we assume the net profit condition be satisfied, and in Section 4 we will prove precisely that $\varphi(u) = 0$ almost always if (6) is not fulfillled. Breach of the net profit condition consists from two options too: $\mathbb{E}X = \kappa$ or $\mathbb{E}X > \kappa$. Therefore, the whole structure of this paper can be seen as based on the expectation of r.v. X: on, shift to the left, or to the right comparing to $\kappa \in \mathbb{N}$.

It is worth mentioning that the exact recursive formulas for the finite time ruin probability $\psi(u,T) := 1 - \varphi(u,T)$ for an even more generalized model than (3), was obtained in [15]. Authors there derive the finite time ruin probability calculation formulas for the model:

$$u + t - \sum_{i=1}^{t} X_i, \; t \in \mathbb{N}$$

supposing $u \in \mathbb{N}_0$ and allowing r.v.s X_i to be a non-negative integer valued and independent, but not necessary identically distributed. Then, by using certain shifts, the model is generalized for certain rational values of initial surplus, premium, and claim sizes $\{X_1, X_2, \ldots\}$. However, similar tricks do not work for the ultimate time ruin probability. We complete the introduction section with the following assertion on a couple of properties of φ which will be often used in the later sections.

Lemma 1. *For the GPDTRM (3) under the net profit condition $\mathbb{E}X < \kappa$ the following relations hold:*

$$\lim_{u\to\infty} \varphi(u) = 1, \tag{7}$$

$$\lim_{v\to\infty} \sum_{i=0}^{v+\kappa} \overline{H}(v+\kappa-i)\varphi(i) = \mathbb{E}X. \tag{8}$$

Proof. The proof of the first property (7) starts with an observation that:

$$\varphi(u) = \mathbb{P}\left(\sup_{n\geqslant 1}\left\{\sum_{i=1}^{n}(X_i-\kappa)\right\} < u\right),$$

and the strong law of large numbers implies that:

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n}(X_i-\kappa)}{n} = \mathbb{E}X - \kappa < 0$$

almost surely. We can now mimic the proof of Theorem 2.3 in [16] and derive that:

$$\liminf_{u\to\infty} \varphi(u) \geqslant 1 - \varepsilon,$$

where ε is an arbitrary small positive number. This implies (7).

The second relation of the lemma follows by an observation that the lower and upper bounds of the sum in (8) are the same. Indeed,

$$\sum_{i=0}^{v+\kappa} \overline{H}(v+\kappa-i)\varphi(i) \leqslant \sum_{i=0}^{v+\kappa} \overline{H}(i) \underset{v\to\infty}{\to} \mathbb{E}X.$$

In addition, for a temporary fixed non-negative M,

$$\sum_{i=0}^{v+\kappa} \overline{H}(v+\kappa-i)\varphi(i) = \left(\sum_{i=0}^{M} + \sum_{i=M+1}^{v+\kappa}\right)\overline{H}(v+\kappa-i)\varphi(i) \geqslant \inf_{i\geqslant M+1}\varphi(i) \sum_{i=0}^{v+\kappa-M-1} \overline{H}(i),$$

where the last term tends to $\inf_{i\geqslant M+1}\varphi(i)\mathbb{E}X$ as $v\to\infty$ and $\inf_{i\geqslant M+1}\varphi(i)$ tends to unit as $M\to\infty$ due to the derived relation (7). Lemma is proved. □

The rest of the paper is organized as follows. In Section 2, the algorithms are presented for the ultimate time survival probability calculation of the GPDTRM with premium rate $\kappa = 2$. Proofs of the main results for the case $\kappa = 2$ are given in Section 6. Additionally, in Section 2, we present one special observation on the limit behavior of a certain recurrent sequence for the model (3) with $\kappa = 1$. Section 3 is dealt with the results on the ultimate time survival probability for model (3) with $\kappa \geqslant 3$. The proofs of these results are presented in Section 7. Section 4 is devoted for the unsatisfied net profit condition and in Section 5 a numerical calculations are given for some theoretical statements illustration.

2. Particular Cases of *GPDTRM*

In this section, we investigate in detail the survival probability φ for model (3) when $\kappa = 2$. In addition, we derive one interesting recurrent tendency to the expectation $\mathbb{E}X$ when $\kappa = 1$.

At first suppose $\kappa = 2$. According to the main model Equation (3) we have that:

$$W(t) = u + 2t - \sum_{i=1}^{t} X_i,\ t \in \mathbb{N}, \tag{9}$$

where $u \in \mathbb{N}_0$ and $X_1, X_2, \ldots$ are independent copies of an integer valued nonnegative r.v. X. Due to (5), the recursive formula of survival probability $\varphi(u)$ is the following:

$$\varphi(u) = \sum_{i=1}^{u+2} h_{u+2-i}\varphi(i) = \sum_{i=0}^{u+1} h_{u-i+1}\varphi(i+1). \tag{10}$$

Below we present theorems that can be used to calculate the ultimate time survival probability for the model (9). The first theorem describes the case $h_0 = \mathbb{P}(X = 0) > 0$.

Theorem 2. *Let us consider the model (9). If $h_0 > 0$ and $\mathbb{E}X < 2$, then*

$$\varphi(0) = \frac{(\varphi(n+1) - \varphi(n)) - (\beta_{n+1} - \beta_n)(2 - \mathbb{E}X)}{\alpha_{n+1} - \alpha_n},\ n \in \mathbb{N}_0, \tag{11}$$

$$\varphi(1) = \frac{1}{h_0}\left(-\varphi(0) + (2 - \mathbb{E}X)\right), \tag{12}$$

$$\varphi(u) = \frac{1}{h_0}\left(\varphi(u-2) - \sum_{k=1}^{u-1} h_{u-i}\varphi(i)\right),\ u \geqslant 2. \tag{13}$$

where sequences α_n and β_n are defined by the following recurrent equalities:

$$\alpha_0 = 1,\ \alpha_1 = -\frac{1}{h_0},\ \alpha_n = \frac{1}{h_0}\left(\alpha_{n-2} - \sum_{i=1}^{n-1} h_{n-i}\alpha_i\right),\ n \geqslant 2,$$

$$\beta_0 = 0,\ \beta_1 = \frac{1}{h_0},\ \beta_n = \frac{1}{h_0}\left(\beta_{n-2} - \sum_{i=1}^{n-1} h_{n-i}\beta_i\right),\ n \geqslant 2.$$

In addition, for each $n \in \mathbb{N}_0$, $\alpha_{n+1} - \alpha_n \neq 0$ and

$$\varphi(n) = \varphi(0)\alpha_n + \beta_n(2 - \mathbb{E}X). \tag{14}$$

Remark 1. *From the definition of the survival probability (4), it is evident that $\varphi(n) \leqslant \varphi(n+1) \leqslant 1$ and $\varphi(n+1) - \varphi(n) \to 0$ as $n \to \infty$. Therefore, for practical calculations we assume that $\varphi(n+1) - \varphi(n) \approx 0$ if n is sufficiently large in Theorem 2.*

Remark 2. *According to Lemma 1, when the net profit condition holds, $\varphi(n) \to 1$ as $n \to \infty$. Therefore, we can get the value of $\varphi(0)$ using equality (14) of Theorem 2. Indeed, inequalities in (23) ensure that $\alpha_n \neq 0$ for all $n \in \mathbb{N}_0$, and consequently the equality (14) implies that $\varphi(0) \approx (\varphi(n) - \beta_n(2 - \mathbb{E}X))/\alpha_n$ for sufficiently large n. It is hard to argue which algorithm is better for finding $\varphi(0)$, however one may think that for some slowly increasing $\varphi(n)$ the assumption $\varphi(n+1) - \varphi(n) \approx 0$ is more accurate than $\varphi(n) \approx 1$. See the Section 5 for more detailed examples on that.*

The net profit condition $\mathbb{E}X < 2$ for the model (9) may remain satisfied if $h_0 = 0$ and $h_1 > 0$. If that happens, the survival probability $\varphi(u)$ for $u \in \mathbb{N}_0$ can be calculated using the following assertion.

Theorem 3. *If $h_0 = 0$ and $\mathbb{E}X < 2$, then:*

$$\varphi(0) = 2 - \mathbb{E}X,\ \varphi(1) = \frac{\varphi(0)}{h_1},\ \varphi(u) = \frac{1}{h_1}\left(\varphi(u-1) - \sum_{i=1}^{u-1} h_{u-i+1}\varphi(i)\right),\ u \geqslant 2.$$

The first required initial values $\varphi(0)$ and $\varphi(1)$ if $h_0 > 0$, or just $\varphi(0)$ if $h_0 = 0$ and $h_1 > 0$, needed for the recursive relation (10), may be calculated a bit differently than in Theorems 2 and 3. This follows from the following assertion.

Theorem 4. *Let us consider the model (9).*
(i) *If $h_0 > 0$ and $\mathbb{E}X < 2$, then:*

$$\begin{pmatrix} \overline{\alpha}_n^{(0)} & \overline{\alpha}_n^{(1)} \\ \overline{\alpha}_{n+1}^{(0)} & \overline{\alpha}_{n+1}^{(1)} \end{pmatrix} \times \begin{pmatrix} \varphi(0) \\ \varphi(1) \end{pmatrix} = \begin{pmatrix} \varphi(n) \\ \varphi(n+1) \end{pmatrix},$$

where

$$\overline{\alpha}_0^{(0)} = 1,\ \overline{\alpha}_1^{(0)} = 0,\ \overline{\alpha}_n^{(0)} = \frac{1}{h_0}\left(\overline{\alpha}_{n-2}^{(0)} - \sum_{i=1}^{n-1} h_{n-i}\overline{\alpha}_i^{(0)}\right),\ n \geqslant 2,$$
$$\overline{\alpha}_0^{(1)} = 0,\ \overline{\alpha}_1^{(1)} = 1,\ \overline{\alpha}_n^{(1)} = \frac{1}{h_0}\left(\overline{\alpha}_{n-2}^{(1)} - \sum_{i=1}^{n-1} h_{n-i}\overline{\alpha}_i^{(1)}\right),\ n \geqslant 2.$$

(ii) *If $h_0 = 0$ and $\mathbb{E}X < 2$, then $\varphi(0) = \varphi(n)/\hat{\alpha}_n$, where:*

$$\hat{\alpha}_0 = 1,\ \hat{\alpha}_1 = \frac{1}{h_1},\ \hat{\alpha}_n = \frac{1}{h_1}\left(\hat{\alpha}_{n-1} - \sum_{i=1}^{n-1} h_{n+1-i}\hat{\alpha}_i\right)$$

with property $1 \leqslant \hat{\alpha}_n \leqslant \hat{\alpha}_{n+1}$ for all $n \in \mathbb{N}_0$.

Remark 3. *The implication of $\varphi(0)$ and $\varphi(1)$, or just $\varphi(1)$, by Theorem 4 is evident in terms that $\varphi(n) \approx 1$ when the net profit condition holds and n is sufficiently large. The remaining values of $\varphi(u)$ when $u \geqslant 2$ are of course implied by (10). However, the efficiency of Theorem 4 is low when compared to Theorems 2 and 3 due to the n size to get a sufficient precision initial values $\varphi(0)$ and $\varphi(1)$ (or $\varphi(0)$ only). See Section 5 for some explicit examples on that.*

Remark 4. *We can not prove that the matrix in the first part of Theorem 4, formed by coefficients $\overline{\alpha}_n^{(0)}$ and $\overline{\alpha}_n^{(1)}$, is non-singular for all $n \in \mathbb{N}_0$. Attempts to prove and calculations with some chosen distributions lead to the following conjecture.*

Let $\overline{D}_n$ denote determinant of matrix in part (i) of Theorem 4, i.e.

$$\overline{D}_n = \overline{\alpha}_n^{(0)}\,\overline{\alpha}_{n+1}^{(1)} - \overline{\alpha}_n^{(1)}\,\overline{\alpha}_{n+1}^{(0)}.$$

Conjecture 1. *For the defined determinants $\overline{D}_n$, it holds that $1 \leqslant \overline{D}_{2n} \leqslant \overline{D}_{2n+2}$ and $-1/h_0 \geqslant \overline{D}_{2n+1} \geqslant \overline{D}_{2n+3}$ for all $n \in \mathbb{N}_0$.*

Comparing Theorems 2 and 3 to Theorem 4, one may observe that defined recurrent sequences have some interesting limit properties. For example $\lim_{n\to\infty} \hat{\alpha}_n = 1/(2 - \mathbb{E}X)$. A simple illustration of that can be obtained also for the discrete time risk model when we set $\kappa = 1$ in (3). It is well known (see, for example, [17–21]) that for the discrete time risk model with $\kappa = 1$

$$W(t) = u + t - \sum_{i=1}^{t} X_i,\ u \in \mathbb{N}_0,\ t \in \mathbb{N},\ X_i \overset{d}{=} X \tag{15}$$

the survival probability for $u = 0$ is $\varphi(0) = 1 - \mathbb{E}X$. On the other hand, from (5) with $\kappa = 1$, we can express $\varphi(0)$ via $\varphi(n)$. Such thoughts lead to the following statement.

Theorem 5. *For the discrete time risk model (15) with the satisfied net profit condition $\mathbb{E}X < 1$, it holds that:*

$$\lim_{n\to\infty} \tilde{\alpha}_n = \frac{1}{1 - \mathbb{E}X},$$

where $\tilde{\alpha}_0 = 1$, $\tilde{\alpha}_1 = 1/h_0$ and $\tilde{\alpha}_n = \frac{1}{h_0}\left(\tilde{\alpha}_{n-1} - \sum_{i=1}^{n-1} h_{n-i}\tilde{\alpha}_i\right)$ for $n \geqslant 2$.

We remind that proofs of statements of this section are given in Section 6 below.

3. General *GPDTRM*s

In this Section, we proceed to develop statements for the ultimate time survival probability calculation for model (3) when $\kappa \in \{3, 4, \ldots\}$. Due to the Remarks 2 and 3 in the previous section on algorithm efficiency, we will not develop any statements on finding initial values for (5) straight forward by (5) itself as in Theorem 4. In addition, we will not gather any initial values without setting differences $\varphi(n+1) - \varphi(n)$, $\varphi(n+2) - \varphi(n)$, and so on. The following three theorems provide us an algorithm to calculate desired values of survival probability.

Theorem 6. *Let us consider the general GPDTRM with $\kappa \geqslant 3$. If $h_0 > 0$ and $\mathbb{E}X < \kappa$ then $\varphi(0), \ldots, \varphi(\kappa-2)$ for all $n \in \mathbb{N}_0$ satisfy the following system of equations:*

$$\begin{pmatrix} \alpha_{n+1}^{(0)} - \alpha_n^{(0)} & \alpha_{n+1}^{(1)} - \alpha_n^{(1)} & \ldots & \alpha_{n+1}^{(\kappa-2)} - \alpha_n^{(\kappa-2)} \\ \alpha_{n+2}^{(0)} - \alpha_n^{(0)} & \alpha_{n+2}^{(1)} - \alpha_n^{(1)} & \ldots & \alpha_{n+2}^{(\kappa-2)} - \alpha_n^{(\kappa-2)} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_{n+\kappa-1}^{(0)} - \alpha_n^{(0)} & \alpha_{n+\kappa-1}^{(1)} - \alpha_n^{(1)} & \ldots & \alpha_{n+\kappa-1}^{(\kappa-2)} - \alpha_n^{(\kappa-2)} \end{pmatrix} \times \begin{pmatrix} \varphi(0) \\ \varphi(1) \\ \ldots \\ \varphi(\kappa-2) \end{pmatrix}$$

$$+ \begin{pmatrix} \tilde{\beta}_{n+1} - \tilde{\beta}_n \\ \tilde{\beta}_{n+2} - \tilde{\beta}_n \\ \ldots \\ \tilde{\beta}_{n+\kappa-1} - \tilde{\beta}_n \end{pmatrix} \times (\kappa - \mathbb{E}X) = \begin{pmatrix} \varphi(n+1) - \varphi(n) \\ \varphi(n+2) - \varphi(n) \\ \vdots \\ \varphi(n+\kappa-1) - \varphi(n) \end{pmatrix}, \tag{16}$$

where coefficients $\alpha_n^{(i)}$ and $\tilde{\beta}_n$ for $n = 0, \ldots, \kappa-1$ are:

n	$\alpha_n^{(0)}$	$\alpha_n^{(1)}$	$\ldots$	$\alpha_n^{(\kappa-2)}$	$\tilde{\beta}_n$
0	1	0	$\ldots$	0	0
1	0	1	$\ldots$	0	0
$\ldots$			$\ldots$		
$\kappa-2$	0	0	$\ldots$	1	0
$\kappa-1$	$-1/h_0$	$-H(\kappa-2)/h_0$	$\ldots$	$-H(1)/h_0$	$1/h_0$

and for $n = \kappa, \kappa + 1, \ldots$

$$\alpha_n^{(i)} = \frac{1}{h_0}\left(\alpha_{n-\kappa}^{(i)} - \sum_{m=1}^{n-1} h_{n-m}\alpha_m^{(i)}\right), i = 0, \ldots, \kappa - 2, \tilde{\beta}_n = \frac{1}{h_0}\left(\tilde{\beta}_{n-\kappa} - \sum_{m=1}^{n-1} h_{n-m}\tilde{\beta}_m\right).$$

In addition,

$$\varphi(\kappa - 1) = \frac{1}{h_0}\left(-\varphi(0) - \sum_{i=1}^{\kappa-2} H(\kappa - 1 - i)\varphi(i) + (\kappa - \mathbb{E}X)\right),$$
$$\varphi(u) = \frac{1}{h_0}\left(\varphi(u - \kappa) - \sum_{i=1}^{u-1} h_{u-i}\varphi(i)\right), u \geqslant \kappa.$$

By the same argumentation as in Remark 1, the right hand side of (16) tends to the zero vector as $n \to \infty$. Therefore, for numerical calculations we assume that the right hand side of (16) is zero vector for some sufficiently large n and obtain $\varphi(0), \ldots, \varphi(\kappa - 2)$ by solving the system. For some chosen distributions and κ we never find the matrix in (16):

$$\begin{pmatrix} \alpha_{n+1}^{(0)} - \alpha_n^{(0)} & \alpha_{n+1}^{(1)} - \alpha_n^{(1)} & \ldots & \alpha_{n+1}^{(\kappa-2)} - \alpha_n^{(\kappa-2)} \\ \alpha_{n+2}^{(0)} - \alpha_n^{(0)} & \alpha_{n+2}^{(1)} - \alpha_n^{(1)} & \ldots & \alpha_{n+2}^{(\kappa-2)} - \alpha_n^{(\kappa-2)} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_{n+\kappa-1}^{(0)} - \alpha_n^{(0)} & \alpha_{n+\kappa-1}^{(1)} - \alpha_n^{(1)} & \ldots & \alpha_{n+\kappa-1}^{(\kappa-2)} - \alpha_n^{(\kappa-2)} \end{pmatrix} \tag{17}$$

to be singular for any $n \in \mathbb{N}_0$. However, to give a strict mathematical proof of that is challenging. The most simple version of (17) is when $\kappa = 3$. In this particular case,

$$\begin{pmatrix} \dot{\alpha}_{n+1}^{(0)} - \dot{\alpha}_n^{(0)} & \dot{\alpha}_{n+1}^{(1)} - \dot{\alpha}_n^{(1)} \\ \dot{\alpha}_{n+2}^{(0)} - \dot{\alpha}_n^{(0)} & \dot{\alpha}_{n+2}^{(1)} - \dot{\alpha}_n^{(1)} \end{pmatrix}, \tag{18}$$

where

$$\dot{\alpha}_0^{(0)} = 1, \dot{\alpha}_1^{(0)} = 0, \dot{\alpha}_2^{(0)} = -\frac{1}{h_0}, \dot{\alpha}_0^{(1)} = 0, \dot{\alpha}_1^{(1)} = 1, \dot{\alpha}_2^{(1)} = -\frac{H(1)}{h_0}.$$

and, for $n \geqslant 3$,

$$\dot{\alpha}_n^{(0)} = \frac{1}{h_0}\left(\dot{\alpha}_{n-3}^{(0)} - \sum_{m=1}^{n-1} h_{n-m}\dot{\alpha}_m^{(0)}\right), \dot{\alpha}_n^{(1)} = \frac{1}{h_0}\left(\dot{\alpha}_{n-3}^{(1)} - \sum_{m=1}^{n-1} h_{n-m}\dot{\alpha}_m^{(1)}\right).$$

Then, for the determinant of matrix (18):

$$\begin{vmatrix} \dot{\alpha}_{n+1}^{(0)} - \dot{\alpha}_n^{(0)} & \dot{\alpha}_{n+1}^{(1)} - \dot{\alpha}_n^{(1)} \\ \dot{\alpha}_{n+2}^{(0)} - \dot{\alpha}_n^{(0)} & \dot{\alpha}_{n+2}^{(1)} - \dot{\alpha}_n^{(1)} \end{vmatrix} = \begin{vmatrix} \dot{\alpha}_{n+1}^{(0)} & \dot{\alpha}_{n+1}^{(1)} \\ \dot{\alpha}_{n+2}^{(0)} & \dot{\alpha}_{n+2}^{(1)} \end{vmatrix} + \begin{vmatrix} \dot{\alpha}_{n}^{(0)} & \dot{\alpha}_{n}^{(1)} \\ \dot{\alpha}_{n+1}^{(0)} & \dot{\alpha}_{n+1}^{(1)} \end{vmatrix} - \begin{vmatrix} \dot{\alpha}_{n}^{(0)} & \dot{\alpha}_{n}^{(1)} \\ \dot{\alpha}_{n+2}^{(0)} & \dot{\alpha}_{n+2}^{(1)} \end{vmatrix}$$
$$=: \dot{D}_{n+1} + \dot{D}_n - \ddot{D}_n.$$

This leads to the following conjecture, which related versions may also be found in [22].

Conjecture 2. *If* $h_0 > 0$ *and* $\kappa \in \{3, 4, \ldots\}$ *then the matrix (17) is non-singular for all* $n \in \mathbb{N}_0$. *In particular, if* $\kappa = 3$, *then* $0 < \dot{D}_n < \dot{D}_{n+1}$ *and* $\ddot{D}_{n+1} < \ddot{D}_n < 0$.

Let us turn to cases when $h_0 = 0$, but the net profit condition $\mathbb{E}X < \kappa$ is still satisfied. One can observe that there are κ distinct versions of such a situation. In addition, one may observe that $\mathbb{E}X \geqslant \kappa$ if $h_0 = \ldots = h_{\kappa-1} = 0$.

Suppose that $h_0 = 0$ and let $l := \min\{1 \leqslant l \leqslant \kappa - 1 : h_l > 0\}$. In other words, we suppose that $h_0 = \ldots = h_{l-1} = 0$ and $h_l > 0$ when $1 \leqslant l \leqslant \kappa - 1$. Then, by Lemma 5 in Section 7, it holds that:

$$\varphi(0) + \sum_{i=1}^{\kappa-1-l} H(\kappa - 1 - i)\varphi(i) = \kappa - \mathbb{E}X. \tag{19}$$

Analogically, from (5) it follows that:

$$\varphi(u) = \sum_{i=1}^{u+\kappa-l} h_{u+\kappa-i}\varphi(i). \tag{20}$$

These two equalities leads to the following assertion on the survival probabilities under requirements $h_0 = 0$ and $\mathbb{E}X < \kappa$.

Theorem 7. *Let us consider the general GPDTRM with $\kappa \geqslant 3$. If $h_0 = \ldots = h_{l-1} = 0$ and $h_l > 0$ when $1 \leqslant l \leqslant \kappa - 2$, and $\mathbb{E}X < \kappa$ then $\varphi(0), \ldots, \varphi(\kappa - l - 2)$ for all $n \in \mathbb{N}_0$ satisfy the following equalities:*

$$\begin{pmatrix} \hat{\alpha}_{n+1}^{(0)} - \hat{\alpha}_n^{(0)} & \hat{\alpha}_{n+1}^{(1)} - \hat{\alpha}_n^{(1)} & \ldots & \hat{\alpha}_{n+1}^{(\kappa-2-l)} - \hat{\alpha}_n^{\kappa-2-l} \\ \hat{\alpha}_{n+2}^{(0)} - \hat{\alpha}_n^{(0)} & \hat{\alpha}_{n+2}^{(1)} - \hat{\alpha}_n^{(1)} & \ldots & \hat{\alpha}_{n+2}^{(\kappa-2-l)} - \hat{\alpha}_n^{(\kappa-2-l)} \\ \ldots & \ldots & \ldots & \ldots \\ \hat{\alpha}_{n+\kappa-1-l}^{(0)} - \hat{\alpha}_n^{(0)} & \hat{\alpha}_{n+\kappa-1-l}^{(1)} - \hat{\alpha}_n^{(1)} & \ldots & \hat{\alpha}_{n+\kappa-1-l}^{(\kappa-2-l)} - \hat{\alpha}_n^{(\kappa-2-l)} \end{pmatrix} \times \begin{pmatrix} \varphi(0) \\ \varphi(1) \\ \ldots \\ \varphi(\kappa-2-l) \end{pmatrix}$$
$$+ \begin{pmatrix} \hat{\beta}_{n+1} - \hat{\beta}_n \\ \hat{\beta}_{n+2} - \hat{\beta}_n \\ \ldots \\ \hat{\beta}_{n+\kappa-1-l} - \hat{\beta}_n \end{pmatrix} \times (\kappa - \mathbb{E}X) = \begin{pmatrix} \varphi(n+1) - \varphi(n) \\ \varphi(n+2) - \varphi(n) \\ \vdots \\ \varphi(n+\kappa-1-l) - \varphi(n) \end{pmatrix}, \tag{21}$$

where coefficients $\hat{\alpha}_n^{(i)}$ and $\hat{\beta}_n$ for $n = 0, \ldots, \kappa - 1 - l$ are:

n	$\hat{\alpha}_n^{(0)}$	$\hat{\alpha}_n^{(1)}$	$\ldots$	$\hat{\alpha}_n^{(\kappa-2-l)}$	$\hat{\beta}_n$
0	1	0	$\ldots$	0	0
1	0	1	$\ldots$	0	0
$\ldots$			$\ldots$		
$\kappa - 2 - l$	0	0	$\ldots$	1	0
$\kappa - 1 - l$	$-1/h_l$	$-H(\kappa-2)/h_l$	$\ldots$	$-H(l+1)/h_l$	$1/h_l$

and for $n = \kappa - l,\ \kappa - l + 1, \ldots$

$$\hat{\alpha}_n^{(i)} = \frac{1}{h_l}\left(\hat{\alpha}_{n-k+l}^{(i)} - \sum_{m=1}^{n-1} h_{n+l-m}\hat{\alpha}_m^{(i)}\right),\ i = 0, \ldots, \kappa - 2 - l,\ \hat{\beta}_n = \frac{1}{h_l}\left(\hat{\beta}_{n-\kappa+l} - \sum_{m=1}^{n-1} h_{n+l-m}\hat{\beta}_m\right).$$

In addition,

$$\varphi(\kappa - l - 1) = \frac{1}{h_l}\left(-\varphi(0) - \sum_{i=1}^{\kappa-2-l} H(\kappa - 1 - i)\varphi(i) + (\kappa - \mathbb{E}X)\right),$$
$$\varphi(u) = \frac{1}{h_l}\left(\varphi(u - \kappa + l) - \sum_{i=1}^{u-1} h_{u+l-i}\varphi(i)\right),\ u \geqslant \kappa - l.$$

Remark 5. *For the quadratic matrix in (21) formed by $\hat{\alpha}_n$ coefficients for $\kappa - 2 - l \geqslant 1$ we can not prove its non-singularity. However, we never find it being singular and conjecture that it is non-singular for any underlying distribution of r.v. X and $\kappa - 2 - l \geqslant 1$.*

Our last statement of this section is a generalized version of Theorem 3.

Theorem 8. *If $h_0 = \ldots = h_{\kappa-2} = 0$ and $h_{\kappa-1} > 0$ and $\mathbb{E}X < \kappa$ then:*

$$\varphi(0) = \kappa - \mathbb{E}X,\ \varphi(1) = \frac{\varphi(0)}{h_{\kappa-1}},\ \varphi(u) = \frac{1}{h_{\kappa-1}}\left(\varphi(u-1) - \sum_{i=1}^{u-1} h_{u-1+\kappa-i}\varphi(i)\right),\ u \geqslant 2.$$

As mentioned, proofs of statements of this section are given in Section 7 below.

4. Survival Probability for *GPDTRM* with the Unsatisfied Net Profit Condition

This section deals with the following statement.

Theorem 9. *Let us consider the general model (3). Then*

- $\varphi(u) = 0$ *for all* $u \in \mathbb{N}_0$ *if* $\mathbb{E}X > \kappa$,
- $\varphi(0) = 0$ *and* $\varphi(u) = 1$ *for all* $u \in \mathbb{N}$ *if* $\mathbb{E}X = \kappa$ *and* $\mathbb{P}(X = \kappa) = 1$,
- $\varphi(u) = 0$ *for all* $u \in \mathbb{N}_0$ *if* $\mathbb{E}X = \kappa$ *and* $\mathbb{P}(X = \kappa) < 1$.

Although the statement of Theorem 9 is known, see for example [23], for the last bullet a certain simplification of proof is possible. Originally the proof of the last bullet was based on certain properties of random walk, see for example [20,24,25], or [23]. In [26], authors simplified the proof avoiding a random walk properties by defining another model, which does satisfy the net profit condition and is as close as we want to the model which it does not. However, this can be simplified even more. Let us turn to the proof of Theorem 9.

Proof. The first assertion of the theorem is implied by the strong law of large numbers. By the same arguments as in the beginning of the proof of Lemma 1 we have $\varphi(u) = \mathbb{P}\left(\sup_{n\geqslant 1}\{\sum_{i=1}^{n}(X_i - \kappa)\} < u\right)$, and $\lim_{n\to\infty}\sum_{i=1}^{n}(X_i-\kappa)/n = \mathbb{E}X - \kappa > 0$ almost surely. Consequently, $\varphi(u) = 0$ for any fixed $u \in \mathbb{N}_0$.

The second option of Theorem 9 is straight forward by definitions (3) and (4).

The last option, $\varphi(u) = 0$ for all $u \in \mathbb{N}_0$ if $\mathbb{E}X = \kappa$ and $\mathbb{P}(X = \kappa) < 1$, is implied by the following logic. Since r.v. X is not degenerate, we can define a random vector (X^*, X), and for X^*, being dependent on X, we enlarge by a certain size the probability of some smaller value and by the same size reduce the probability of some larger value. This leads to the satisfied net profit condition and X^*, being as close as we want to X, does the rest. Such a described proof can be found in [22] for a different model than what is investigated in this paper. Below we present details of the described proof.

Since r.v. X is not degenerate, it has at least two values $i < j$ with positive probabilities. Let us define the integer-valued and non-negative vector (X^*, X) by the following table:

$X^*\backslash X$	0	1	$\ldots$	i	$\ldots$	j	$\ldots$	Σ
0	h_0	0	$\ldots$	0	$\ldots$	0	$\ldots$	h_0
1	0	h_1	$\ldots$	0	$\ldots$	0	$\ldots$	h_1
$\ldots$			$\ldots$		$\ldots$		$\ldots$	$\ldots$
i	0	0	$\ldots$	h_i	$\ldots$	ε/j	$\ldots$	$h_i + \varepsilon/j$
$\ldots$			$\ldots$		$\ldots$		$\ldots$	$\ldots$
j	0	0	$\ldots$	0	$\ldots$	$h_j - \varepsilon/j$	$\ldots$	$h_j - \varepsilon/j$
$\ldots$			$\ldots$		$\ldots$		$\ldots$	$\ldots$
Σ	h_0	h_1	$\ldots$	h_i	$\ldots$	h_j	$\ldots$	1

where ε is a sufficiently small positive number.

We have that $\mathbb{E}X^* = \mathbb{E}X - \varepsilon_1 = \kappa - \varepsilon_1$, where $\varepsilon_1 := \varepsilon(1 - i/j) > 0$. If model (3) is generated by X^*, then this model satisfies the net profit condition $\mathbb{E}X^* - \kappa < 0$ and all statements from Sections 2 and 3 hold for such a model. It is evident that $\mathbb{P}(X^* \leqslant X) = 1$. In addition,

$$\mathbb{P}\left(X^* + X^* \leqslant X + X\right) = \sum_{m=0}^{\infty}\sum_{l=0}^{\infty} \mathbb{P}(X_1^* + m \leqslant X_1 + l)\mathbb{P}(X_2^* = m, X_2 = l)$$
$$= \sum_{m=0,\, m\neq j}^{\infty} \mathbb{P}(X_1^* + m \leqslant X_1 + m)h_m + \mathbb{P}(X_1^* + j \leqslant X_1 + j)\left(h_j - \frac{\varepsilon}{j}\right) + \mathbb{P}(X_1^* + i \leqslant X_1 + j)\frac{\varepsilon}{j} = 1.$$

Assuming that $\mathbb{P}\left(\sum_{m=1}^{n} X_m^* \leqslant \sum_{m=1}^{n} X_m\right) = 1$ it follows:

$$\mathbb{P}\left(\sum_{m=1}^{n+1} X_m^* \leqslant \sum_{m=1}^{n+1} X_m\right) = \sum_{m=0,\, m\neq j}^{\infty} \mathbb{P}\left(\sum_{m=1}^{n} X_m^* \leqslant \sum_{m=1}^{n} X_m\right) h_m + \mathbb{P}\left(\sum_{m=1}^{n} X_m^* \leqslant \sum_{m=1}^{n} X_m\right)\left(h_j - \frac{\varepsilon}{j}\right)$$
$$+ \mathbb{P}\left(\sum_{m=1}^{n} X_m^* + i \leqslant \sum_{m=1}^{n} X_m + j\right)\frac{\varepsilon}{j} = 1.$$

Therefore, for all $u \in \mathbb{N}_0$,

$$\varphi^*(u) := \mathbb{P}\left(\sup_{n\geqslant 1}\left\{\sum_{m=1}^{n}(X_m^* - \kappa)\right\} < u\right) \geqslant \mathbb{P}\left(\sup_{n\geqslant 1}\left\{\sum_{m=1}^{n}(X_m - \kappa)\right\} - u < 0\right) = \varphi(u).$$

The remaining part of the proof depends on a distribution of X. If, for example, $\kappa = 2$, $\mathbb{P}(X^* = 0) = 0$, and $\mathbb{P}(X^* = 1) > 0$, then Theorem 3 gives that $0 \leqslant \varphi(0) \leqslant \varphi^*(0) = \varepsilon_1$. This implies $\varphi(0) = 0$ due to ε_1 being arbitrary small. The remaining equalities $\varphi(u) = 0$ for $u = 1, 2, \ldots$, are implied by Theorem 3. The same way of proof could be easily repeated in view of Theorem 2 if $\kappa = 2$ and $\mathbb{P}(X^* = 0) > 0$. If $\kappa = 3$ and $\mathbb{P}(X^* = 0) > 0$, then by Theorem 6, we get that:

$$\begin{pmatrix} \dot{\alpha}_{n+1}^{(0)} - \dot{\alpha}_n^{(0)} & \dot{\alpha}_{n+1}^{(1)} - \dot{\alpha}_n^{(1)} \\ \dot{\alpha}_{n+2}^{(0)} - \dot{\alpha}_n^{(0)} & \dot{\alpha}_{n+2}^{(1)} - \dot{\alpha}_n^{(1)} \end{pmatrix}\begin{pmatrix} \varphi^*(0) \\ \varphi^*(1) \end{pmatrix} + \begin{pmatrix} \tilde{\beta}_{n+1} - \tilde{\beta}_n \\ \tilde{\beta}_{n+2} - \tilde{\beta}_n \end{pmatrix} \times \varepsilon_1 = \begin{pmatrix} \varphi(n+1) - \varphi(n) \\ \varphi(n+2) - \varphi(n) \end{pmatrix}. \tag{22}$$

It is clear that $0 \leqslant \varphi(0) \leqslant \varphi^*(0)$, $0 \leqslant \varphi(1) \leqslant \varphi^*(1)$ and $(\varphi^*(0), \varphi^*(1))$ is as close to the point $(0, 0)$ as we want if n is sufficiently large in (22) and ε_1 gets closer to 0. The further consideration is similar to the case $\kappa = 2$. □

5. Numerical Examples

First we give numerical examples for statements in Section 2. We compare $\varphi(0)$ calculations by Formulas (11) and (14) of Theorem 2 then turn to Theorem 4. Let us introduce the following notations:

$$\varphi_n^{**}(0) := \frac{1 - \beta_n(2 - \mathbb{E}X)}{\alpha_n},\ \varphi_n^*(0) := \frac{-(\beta_{n+1} - \beta_n)(2 - \mathbb{E}X)}{\alpha_{n+1} - \alpha_n},$$

where coefficients α_n and β_n are defined in Theorem 2. Since $\varphi(n) \leqslant 1$ and $\varphi(n) \leqslant \varphi(n+1)$, Theorem 2 implies:

$$\alpha_n\varphi(0) \leqslant 1 - \beta_n(2 - \mathbb{E}X),\ (\alpha_{n+1} - \alpha_n)\varphi(0) \leqslant -(\beta_{n+1} - \beta_n)(2 - \mathbb{E}X).$$

Therefore, by setting $\Delta_n^{**} := |\varphi_{n+1}^{**}(0) - \varphi_n^{**}(0)|$, $\Delta_n^* := |\varphi_{n+1}^*(0) - \varphi_n^*(0)|$ we can calculate the absolute difference of lower and upper bounds for $\varphi(0)$ estimate depending on n.

We note that in the below tables, where $\varphi(u, T)$ and $\varphi(u)$ are present, we limit u up to 50 or up to some lower number if rounded finite time survival probability equals 1. We also limit a variety of T when some rounded values present no difference.

Example 1. *We say that a r.v. X is* geometrically distributed *with parameter $0 < p < 1$ and denote by $X \sim \mathcal{G}(p)$ if $\mathbb{P}(X = m) = p(1-p)^m$, $m \in \{0, 1, \ldots\}$. Let us consider the model (9) generated by r.v. $X \sim \mathcal{G}\left(\frac{101}{300}\right)$.*

It is clear that the model satisfies the net profit condition $\mathbb{E}X = \frac{199}{101} < 2$, and we can fill Table 1 of approximate values for $\varphi(0)$ by rounding the numbers up to 15 decimal places.

Table 1. Limit tendency to $\varphi(0)$ with given r.v.

n	$\varphi_n^{**}(0)$	$\varphi_n^{*}(0)$	Δ_n^{**}	Δ_n^{*}
0	−0.306963696369636	0.022221673538925	0.521213979113341	0.003671257796475
1	0.214250282743705	0.018550415742450	0.282622680978633	0.001825335523813
2	−0.068372398234928	0.020375751266263	0.134024426188439	0.000908391492999
3	0.065652027953512	0.019467359773264	0.068195457784529	0.000451859303659
4	−0.002543429831018	0.019919219076923	0.033538955167044	0.000224818998900
5	0.030995525336026	0.019694400078023	0.016780308320236	0.000111844141051
6	0.014215217015790	0.019806244219074	0.008324674777041	0.000055643971685
7	0.022539891792831	0.019750600247389	0.004147376229806	0.000027682846108
8	0.018392515563025	0.019778283093497	0.002061894240081	0.000013772394614
9	0.020454409803106	0.019764510698883	0.001026157583478	0.000006851807318
10	0.019428252219628	0.019771362506201	0.000510429310434	0.000003408806392
20	0.019768769544779	0.019769088294605	0.000000474162782	0.000000003166424
30	0.019769085886016	0.019769086182101	0.000000000440448	0.000000000002941
40	0.019769086179864	0.019769086180139	0.000000000000409	0.000000000000003
50	0.019769086180137	0.019769086180137	0	0
100	0.019769086180137	0.019769086180137	0	0

From Table 1 it is easy to see that $\Delta_n^{**} > \Delta_n^{*}$ (except when we compare zeros) for all n being present in that table. Solutions of the system in Theorem 4 are:

$$(0.031064536986934, 0.046365746949288) \text{ if } n = 100,$$
$$(0.019769986553721, 0.029507930349202) \text{ if } n = 1000,$$

where the first component tends to $\varphi(0)$. However, the convergence is much slower with respect to that in the previous table. This example illustrates that Theorem 2 is more efficient to estimate $\varphi(0)$. In Table 2 we present more values of survival probability of finite and ultimate time for model (9) with $X \sim \mathcal{G}\left(\frac{101}{300}\right)$ according to Theorems 1 and 2. All values in Table 2 are rounded up to 3 decimal places.

Table 2. Finite and ultimate time survival probability for Example 1.

$T \backslash u$	0	1	2	3	4	5	10	20	30	40	50
1	0.560	0.708	0.806	0.872	0.915	0.943	0.993	1	1	1	1
2	0.430	0.578	0.692	0.776	0.839	0.885	0.980	0.999	1	1	1
3	0.362	0.502	0.615	0.706	0.777	0.833	0.964	0.999	1	1	1
4	0.319	0.449	0.560	0.652	0.727	0.788	0.946	0.997	1	1	1
5	0.289	0.411	0.517	0.608	0.685	0.749	0.927	0.996	1	1	1
6	0.266	0.381	0.484	0.573	0.650	0.715	0.909	0.994	1	1	1
7	0.248	0.357	0.456	0.543	0.620	0.686	0.890	0.991	0.999	1	1
8	0.233	0.338	0.433	0.518	0.593	0.659	0.873	0.988	0.999	1	1
9	0.221	0.321	0.413	0.496	0.570	0.636	0.856	0.984	0.999	1	1
10	0.211	0.307	0.395	0.476	0.550	0.615	0.839	0.981	0.998	1	1
20	0.153	0.226	0.295	0.361	0.423	0.481	0.713	0.933	0.988	0.998	1
30	0.127	0.188	0.247	0.304	0.359	0.411	0.631	0.882	0.971	0.994	0.999
40	0.112	0.166	0.218	0.269	0.318	0.366	0.573	0.836	0.949	0.987	0.997
50	0.101	0.150	0.198	0.245	0.290	0.334	0.529	0.796	0.926	0.977	0.994
∞	0.020	0.030	0.039	0.049	0.058	0.067	0.113	0.197	0.273	0.342	0.405

Example 2. *We say that a r.v.* X *is the* shifted version of geometric distribution *with parameter* $0 < p < 1$ *(denote by* $X \sim \mathcal{G}_{shift}(p)$*) if* $\mathbb{P}(X = m) = p(1-p)^{m-1}$, $m \in \{1, 2, \ldots\}$*. Let* $X \sim \mathcal{G}_{shift}\left(\frac{101}{200}\right)$ *and let us consider the model (9) with the satisfied net profit condition* $\mathbb{E}X = \frac{200}{101} < 2$.

The advantage of finding $\varphi(0)$ by Theorem 3 against the equality of part (ii) in Theorem 4 is not questionable. Theorem 4 gives the results being presented in Table 3.

Table 3. Approximations of $\varphi(0)$ in Example 2.

n	0	1	2	3	4	5	50	300	1000
$1/\hat{\alpha}_n$	1	0.5050	0.3400	0.2575	0.2081	0.1751	0.0310	0.0199	0.0198

In Table 4, values of survival probability (finite and ultimate time) are presented for the model in Example 2. To calculate these values we used Theorem 1 and Theorem 3. All values are rounded up to 3 decimal places.

Table 4. Finite and ultimate time survival probabilities for the model in Example 2.

$T \backslash u$	0	1	2	3	4	5	10	20	30	40	50
1	0.505	0.755	0.879	0.940	0.970	0.985	1.000	1	1	1	1
2	0.381	0.632	0.788	0.880	0.933	0.963	0.998	1	1	1	1
3	0.319	0.556	0.720	0.827	0.896	0.938	0.996	1	1	1	1
4	0.281	0.502	0.666	0.782	0.861	0.913	0.993	1	1	1	1
5	0.254	0.462	0.624	0.744	0.829	0.888	0.989	1	1	1	1
7	0.217	0.405	0.559	0.681	0.774	0.843	0.980	1	1	1	1
9	0.194	0.365	0.512	0.632	0.728	0.802	0.968	1	1	1	1
10	0.185	0.350	0.492	0.611	0.708	0.784	0.962	0.999	1	1	1
20	0.134	0.260	0.375	0.479	0.571	0.651	0.894	0.995	1	1	1
30	0.112	0.218	0.318	0.410	0.494	0.570	0.832	0.985	0.999	1	1
40	0.098	0.193	0.282	0.366	0.444	0.515	0.780	0.972	0.998	1	1
50	0.089	0.175	0.257	0.334	0.407	0.475	0.737	0.956	0.995	1	1
∞	0.020	0.030	0.039	0.049	0.058	0.067	0.113	0.197	0.273	0.342	0.405

We now turn to illustrations of the statements from Section 3. For this we present three additional examples.

Example 3. *Let* $X \sim \mathcal{G}\left(\frac{101}{300}\right)$ *again as in Example 1 and let us consider the model (3) with* $\kappa = 3$.

Due to Theorems 1 and 6 we fill Table 5 with the values of $\varphi(u)$ by rounding values of survival probability up to 3 decimal places.

Table 5. Finite and ultimate time survival probabilities for the model in Example 3.

$T \backslash u$	0	1	2	3	4	5	10	20	30	40
1	0.708	0.806	0.872	0.915	0.943	0.963	0.995	1	1	1
2	0.622	0.730	0.808	0.865	0.905	0.933	0.989	1	1	1
3	0.580	0.689	0.771	0.833	0.878	0.911	0.983	0.999	1	1
4	0.555	0.663	0.747	0.811	0.859	0.895	0.978	0.999	1	1
5	0.538	0.646	0.730	0.795	0.845	0.883	0.973	0.999	1	1
7	0.518	0.624	0.708	0.774	0.825	0.865	0.965	0.998	1	1
9	0.506	0.611	0.695	0.761	0.813	0.854	0.959	0.997	1	1
10	0.502	0.607	0.690	0.756	0.809	0.850	0.957	0.997	1	1
20	0.485	0.588	0.670	0.736	0.789	0.832	0.946	0.995	1	1
30	0.482	0.584	0.666	0.732	0.785	0.827	0.943	0.994	0.999	1

Table 5. *Cont.*

$T\backslash u$	0	1	2	3	4	5	10	20	30	40
40	0.481	0.583	0.665	0.731	0.784	0.826	0.942	0.994	0.999	1
50	0.480	0.582	0.664	0.730	0.783	0.826	0.942	0.993	0.999	1
∞	0.480	0.582	0.664	0.730	0.783	0.825	0.941	0.993	0.999	1

Example 4. *We say that a r.v. X follows the Pascal distribution with parameters $0 < p < 1$ and natural number N (denote by $X \sim \mathcal{P}(N,p)$) if:*

$$\mathbb{P}(X = m) = \binom{m-1}{N-1} p^N (1-p)^{m-N},\ m \in \{N,\ N+1,\ldots\}$$

Parameter N in Pascal distribution describes a required number of successes performing an independent experiments with success probability p each. Let us consider the model (3) with $\kappa = 8$ and $X \sim \mathcal{P}(4, 3/5)$.

Then we have that $\mathbb{E}X = 6 + 2/3 < 8$ and, according to Theorems 1 and 7 we can fulfilll the Table 6 with the values of $\varphi(u)$ (rounded up to 3 decimal places as usual).

Table 6. Finite and ultimate time survival probabilities for the model in Example 4.

$T\backslash u$	0	1	2	3	4	5	10	18
1	0.710	0.826	0.901	0.945	0.971	0.985	1	1
2	0.646	0.770	0.857	0.914	0.949	0.971	0.999	1
3	0.618	0.744	0.834	0.895	0.935	0.961	0.997	1
4	0.603	0.729	0.820	0.884	0.926	0.954	0.996	1
5	0.594	0.720	0.812	0.876	0.920	0.949	0.995	1
10	0.579	0.703	0.796	0.862	0.908	0.939	0.993	1
20	0.575	0.699	0.792	0.858	0.904	0.936	0.992	1
∞	0.575	0.699	0.791	0.858	0.904	0.935	0.991	1

Example 5. *Let us consider the GPDTRM with $\kappa = 8$ and $X \sim \mathcal{P}(7, 22/25)$.*

In the case, we have $\mathbb{E}X = 7.95\ldots < 8$. Therefore, according to Theorems 1 and 8 we fulfill Table 7 with the values of $\varphi(u)$.

Table 7. Survival probability for model in Example 5.

$T\backslash u$	0	1	2	3	4	5	10	20	30	40	50
1	0.409	0.752	0.917	0.976	0.994	0.999	1	1	1	1	1
2	0.307	0.633	0.838	0.937	0.978	0.993	1	1	1	1	1
3	0.259	0.559	0.775	0.897	0.957	0.983	1	1	1	1	1
4	0.229	0.509	0.725	0.860	0.934	0.971	1	1	1	1	1
10	0.155	0.366	0.556	0.706	0.813	0.886	0.995	1	1	1	1
20	0.118	0.283	0.442	0.579	0.691	0.778	0.971	1	1	1	1
30	0.101	0.245	0.386	0.512	0.619	0.707	0.940	0.999	1	1	1
40	0.091	0.222	0.352	0.469	0.571	0.658	0.910	0.998	1	1	1
50	0.085	0.206	0.327	0.438	0.536	0.621	0.883	0.995	1	1	1
∞	0.045	0.111	0.179	0.242	0.301	0.356	0.570	0.809	0.915	0.962	0.983

An impact of r.v. X to survival probabilities $\varphi(u, T)$ and $\varphi(u)$ is well seen when comparing the last two tables. Roughly, the closer $\mathbb{E}X$ to κ is, the lower values of survival probabilities we get.

6. Proofs of Theorems 2–5

The proofs of theorems we begin with the auxiliary lemmas.

Lemma 2. *Let us consider the model (9). If $\mathbb{E}X < 2$, then $\varphi(0) + h_0\varphi(1) = 2 - \mathbb{E}X$.*

Proof. Summing up the both sides of (10) by u from 0 up to some sufficiently large positive integer v, we get:

$$\sum_{u=0}^{v} \varphi(u) = \sum_{u=0}^{v} \sum_{i=0}^{u+1} h_{u-i+1}\varphi(i+1) = \sum_{i=0}^{0} \sum_{u=0}^{v} h_{u-i+1}\varphi(i+1) + \sum_{i=1}^{v+1} \sum_{u=i-1}^{v} h_{u-i+1}\varphi(i+1)$$
$$= (H(v+1) - h_0)\varphi(1) + \sum_{i=0}^{v} H(v-i)\varphi(i+2).$$

Therefore,

$$\sum_{i=0}^{v+2} \varphi(i)\overline{H}(v-i+2) - \varphi(v+1) - \varphi(v+2) = (H(v+1) - h_0)\varphi(1) - H(v+2)\varphi(0) - H(v+1)\varphi(1).$$

Due to Lemma 1 and condition $\mathbb{E}X < 2$, by supposing $v \to \infty$, we derive from the last equation that:

$$\varphi(0) + h_0\varphi(1) = 2 - \mathbb{E}X.$$

□

By setting $u = 0, 1 \ldots, n-2$ into (10) we may get the following sequence of equations:

$$\begin{aligned}
&\varphi(0) - h_1\varphi(1) - h_0\varphi(2) = 0,\\
&\varphi(1)(1 - h_2) - h_1\varphi(2) - h_0\varphi(3) = 0,\\
&\vdots\\
&\varphi(n-2) - \sum_{i=1}^{n} h_{n-i}\varphi(i) = 0.
\end{aligned}$$

This, together with Lemma 2, allows us to express $\varphi(0)$ via $\varphi(n)$, $n \in \mathbb{N}_0$. Such an expression is presented in the following lemma.

Lemma 3. *Let us consider the model (9). If $h_0 > 0$ and $\mathbb{E}X < 2$, then,*

$$\varphi(n) = \alpha_n\varphi(0) + \beta_n(2 - \mathbb{E}X),\ n \in \mathbb{N}_0,$$

where sequences α_n and β_n are defined in Theorem 2

Proof. We use induction. For $n = 0, 1, 2$ the statement is obvious or follows by (10) and Lemma 2. For $n = 3, 4, \ldots$, by (10) and the induction hypothesis, it follows that:

$$\begin{aligned}
\varphi(n+1) &= \frac{1}{h_0}\left(\varphi(n-1) - \sum_{i=1}^{n} h_{n+1-i}\varphi(i)\right)\\
&= \frac{1}{h_0}\left(\alpha_{n-1}\varphi(0) + \beta_{n-1}(2 - \mathbb{E}X) - \sum_{k=1}^{n} h_{n+1-i}\left(\alpha_i\varphi(0) + \beta_i(2 - \mathbb{E}X)\right)\right)\\
&= \frac{1}{h_0}\left(\varphi(0)\left(\alpha_{n-1} - \sum_{i=1}^{n} h_{n+1-i}\alpha_i\right) + (2 - \mathbb{E}X)\left(\beta_{n-1} - \sum_{i=1}^{n} h_{n+1-i}\beta_i\right)\right)\\
&= \alpha_{n+1}\varphi(0) + \beta_{n+1}(2 - \mathbb{E}X).
\end{aligned}$$

□

Lemma 4. *For coefficients α_n defined in Theorem 2 it holds that $|\alpha_{n+1} - \alpha_n| \geqslant 2$ for all $n \in \mathbb{N}_0$.*

Proof. Let us observe that the statement follows from inequalities:

$$\alpha_{2n+2} \geqslant \alpha_{2n} \geqslant 1 \text{ and } \alpha_{2n+3} \leqslant \alpha_{2n+1} \leqslant -1,\ n \in \mathbb{N}_0. \tag{23}$$

Indeed, if n is even, then $n+1$ is odd and:

$$|\alpha_{n+1} - \alpha_n| = \alpha_n - \alpha_{n+1} \geqslant 2.$$

And conversely, if n is odd, then $n+1$ is even and:

$$|\alpha_{n+1} - \alpha_n| = \alpha_{n+1} - \alpha_n \geqslant 2.$$

It remains to show that inequalities in (23) are correct. For this, we use induction. For $n = 0$ we have that:

$$\alpha_2 = \frac{1}{h_0}\left(1 + \frac{h_1}{h_0}\right) \geqslant 1 = \alpha_0,$$

and

$$\alpha_3 = \frac{1}{h_0}\left(\alpha_1 - h_2\alpha_1 - h_1\alpha_2\right) \leqslant -\frac{1}{h_0}\left(\frac{1 + h_1 - h_2}{h_0}\right) \leqslant -\frac{1}{h_0} = \alpha_1 \leqslant -1.$$

In general,

$$\begin{aligned}
\alpha_{2n} &= \frac{1}{h_0}\left(\alpha_{2n-2} - \sum_{i=1}^{2n-1} h_{2n-i}\alpha_i\right) = \frac{1}{h_0}\left(\alpha_{2n-2} - \sum_{i=1}^{2n-1} h_i\alpha_{2n-i}\right) \\
&= \frac{1}{h_0}(\alpha_{2n-2} - (h_1\alpha_{2n-1} + h_3\alpha_{2n-3} + \ldots + h_{2n-1}\alpha_1) - (h_2\alpha_{2n-2} + h_4\alpha_{2n-4} + \ldots + h_{2n-2}\alpha_2)) \\
&\geqslant \frac{1}{h_0}\left(\alpha_{2n-2} + (h_1 + h_3 + \ldots + h_{2n-3}) - \alpha_{2n-2}(h_2 + h_4 + \ldots + h_{2n-2})\right) \\
&\geqslant \frac{\alpha_{2n-2}(1 - h_1 - h_2 - \ldots)}{h_0} = \alpha_{2n-2}.
\end{aligned}$$

and

$$\begin{aligned}
\alpha_{2n+1} &= \frac{1}{h_0}\left(\alpha_{2n-1} - \sum_{i=1}^{2n} h_{2n+1-i}\alpha_i\right) = \frac{1}{h_0}\left(\alpha_{2n-1} - \sum_{i=1}^{2n} h_i\alpha_{2n+1-i}\right) \\
&= \frac{1}{h_0}(\alpha_{2n-1} - (h_1\alpha_{2n} + h_3\alpha_{2n-2} + \ldots + h_{2n-1}\alpha_2) - (h_2\alpha_{2n-1} + h_4\alpha_{2n-3} + \ldots + h_{2n}\alpha_1)) \\
&\leqslant \frac{1}{h_0}\left(\alpha_{2n-1} - (h_1 + h_3 + \ldots + h_{2n-1}) - \alpha_{2n-1}(h_2 + h_4 + \ldots + h_{2n})\right) \\
&\leqslant \frac{\alpha_{2n-1}(1 - h_1 - h_2 - \ldots)}{h_0} = \alpha_{2n-1}.
\end{aligned}$$

□

We note that a similar technique as in Lemmas 3 and 4, but for a different model, was used in [16]. We now prove all theorems from Section 2.

Proof of Theorem 2. The equality (14) follows from Lemma 3 directly, and the expression (11) for $\varphi(0)$ is implied by Lemma 3 by setting the difference $\varphi(n+1) - \varphi(n)$. Lemma 4 ensures that there is no division by zero. Two remaining expressions of $\varphi(1)$ and $\varphi(u)$, $u \geqslant 2$ follow by Lemma 2 and Equation (10) accordingly. □

Proof of Theorem 3. is straightforward in view of Lemma 2 and (10). □

Proof of Theorem 4. The equality (i) of Theorem 4 is evident if $n = 0$. The general case of this equality for an arbitrary non-negative n follows by a similar arguments as in Lemma 3, where we expressed $\varphi(0)$ via $\varphi(n)$. Let us show that:

$$\varphi(n) = \overline{\alpha}_n^{(0)}\varphi(0) + \overline{\alpha}_n^{(1)}\varphi(1).$$

By (10), we obtain:

$$\begin{aligned}
\varphi(n+1) &= \frac{1}{h_0}\left(\varphi(n-1) - \sum_{i=1}^{n} h_{n+1-i}\varphi(i)\right) \\
&= \frac{1}{h_0}\left(\overline{\alpha}_{n-1}^{(0)}\varphi(0) + \overline{\alpha}_{n-1}^{(1)}\varphi(1) - \sum_{i=1}^{n} h_{n+1-i}\left(\overline{\alpha}_i^{(0)}\varphi(0) + \overline{\alpha}_i^{(1)}\varphi(1)\right)\right) \\
&= \frac{1}{h_0}\left(\varphi(0)\left(\overline{\alpha}_{n-1}^{(0)} - \sum_{i=1}^{n} h_{n+1-i}\overline{\alpha}_i^{(0)}\right) + \varphi(1)\left(\overline{\alpha}_{n-1}^{(1)} - \sum_{i=1}^{n} h_{n+1-i}\overline{\alpha}_i^{(1)}\right)\right) \\
&= \overline{\alpha}_{n+1}^{(0)}\varphi(0) + \overline{\alpha}_{n+1}^{(1)}\varphi(1).
\end{aligned}$$

The equality (ii) of Theorem 4 follows also by (10) and induction for $n \in \mathbb{N}_0$,

$$h_1\varphi(n+1) = \varphi(n) - \sum_{i=1}^{n} h_{n+2-i}\varphi(i) = \hat{\alpha}_n\varphi(0) - \sum_{i=1}^{n} h_{n+2-i}\hat{\alpha}_i\varphi(i) = h_1\hat{\alpha}_{n+1}\varphi(0).$$

It remains to show that $\hat{\alpha}_n \neq 0$, $n \in \mathbb{N}_0$. This follows immediately by inequalities $1 \leqslant \hat{\alpha}_n \leqslant \hat{\alpha}_{n+1}$ which can be derived by mathematical induction again:

$$\hat{\alpha}_{n+1} = \frac{1}{h_1}\left(\hat{\alpha}_n - \sum_{i=1}^{n} h_{n+2-i}\hat{\alpha}_i\right) \geqslant \frac{1}{h_1}\left(\hat{\alpha}_n - \hat{\alpha}_n\sum_{i=1}^{n} h_{n+2-i}\right) \geqslant \hat{\alpha}_n \geqslant 1,\ n \in \mathbb{N}_0.$$

□

7. Proofs of Theorems 6–8.

We start with an analog of Lemma 2 which relates $\varphi(0), \ldots, \varphi(k-1)$ values.

Lemma 5. *Let us consider the general model (3) with $\kappa \geqslant 3$. If $\mathbb{E}X < \kappa$ then:*

$$\varphi(0) + \sum_{i=1}^{\kappa-1} H(\kappa-1-i)\varphi(i) = \kappa - \mathbb{E}X.$$

Proof. We sum up the both sides of (5) by u from 0 up to some sufficiently large non-negative integer v

$$\begin{aligned}
\sum_{u=0}^{v}\varphi(u) &= \sum_{u=0}^{v}\sum_{i=1}^{u+\kappa} h_{u+\kappa-i}\varphi(i) = \sum_{i=1}^{\kappa-1}\sum_{u=0}^{v} h_{u+\kappa-i}\varphi(i) + \sum_{i=\kappa}^{v+\kappa}\sum_{u=i-\kappa}^{v} h_{u+\kappa-i}\varphi(i) \\
&= \sum_{i=1}^{\kappa-1}\left(H(v+\kappa-i) - H(\kappa-i-1)\right)\varphi(i) + \sum_{i=\kappa}^{v+\kappa} H(v+\kappa-i)\varphi(i) \\
&= \sum_{i=1}^{v+\kappa} H(v+\kappa-i)\varphi(i) - \sum_{i=1}^{\kappa-1} H(\kappa-i-1)\varphi(i).
\end{aligned}$$

From this

$$\sum_{i=0}^{v+\kappa} \overline{H}(v+\kappa-i)\varphi(i) - \sum_{i=v+1}^{v+\kappa} \varphi(i) = -H(v+\kappa)\varphi(0) - \sum_{i=1}^{\kappa-1} H(\kappa-i-1)\varphi(i).$$

If $\mathbb{E}X < \kappa$ and $v \to \infty$, in view of Lemma 1, we get the desired result from the last equation. □

The same way as we expressed $\varphi(0)$ via $\varphi(n)$ in Lemma 3, we now express $\varphi(0), \ldots, \varphi(\kappa-2)$ via $\varphi(n)$ for $n \in \mathbb{N}_0$. This, in turn, is dependent on the first non-negative value of r.v. X which occurs with a positive probability. First, we consider the case $h_0 > 0$.

Lemma 6. *Suppose that $h_0 > 0$ and $\mathbb{E}X < \kappa$ in the model (3) with $\kappa \geqslant 3$. Then for all $n \in \mathbb{N}_0$*

$$\varphi(n) = \sum_{i=0}^{\kappa-2} \alpha_n^{(i)} \varphi(i) + \tilde{\beta}_n(\kappa - \mathbb{E}X),$$

where coefficients $\alpha_n^{(i)}$ and $\tilde{\beta}_n$ are defined in Theorem 6.

Proof. The statements follows by induction. For $n = 0, \ldots, \kappa-2$ it is obvious and follows by Lemma 5 for $n = \kappa - 1$. Let $n \geqslant \kappa$. Then, by (5) and induction hypothesis we have:

$$\begin{aligned}
\varphi(n+1) &= \frac{1}{h_0}\left(\varphi(n+1-\kappa) - \sum_{m=1}^{n} h_{n+1-m}\varphi(m)\right) \\
&= \frac{1}{h_0}\left(\sum_{i=0}^{\kappa-2} \alpha_{n+1-\kappa}^{(i)}\varphi(i) + \tilde{\beta}_{n+1-\kappa}(\kappa-\mathbb{E}X) - \sum_{m=1}^{n} h_{n+1-m}\left(\sum_{i=0}^{\kappa-2}\alpha_m^{(i)}\varphi(i) + \tilde{\beta}_m(\kappa-\mathbb{E}X)\right)\right) \\
&= \frac{1}{h_0}\left(\sum_{i=0}^{\kappa-2}\varphi(i)\left(\alpha_{n+1-\kappa}^{(i)} - \sum_{m=1}^{n} h_{n+1-m}\alpha_m^{(i)}\right) + (\kappa-\mathbb{E}X)\left(\tilde{\beta}_{n+1-\kappa} - \sum_{m=1}^{n} h_{n+1-m}\tilde{\beta}_m\right)\right) \\
&= \sum_{i=0}^{\kappa-2}\alpha_{n+1}^{(i)}\varphi(i) + \tilde{\beta}_{n+1}(\kappa-\mathbb{E}X). \qquad \square
\end{aligned}$$

□

Proof of Theorem 6. follows now from Lemma 6 and recurrence relation (5). □

Lemma 7. *Let us consider the general model (3) with $\kappa \geqslant 3$. If $h_0 = \ldots = h_{l-1} = 0$ and $h_l > 0$ for some $1 \leqslant l \leqslant \kappa-2$, and $\mathbb{E}X < \kappa$ then:*

$$\varphi(n) = \sum_{i=0}^{\kappa-2-l} \hat{\alpha}_n^{(i)}\varphi(i) + \hat{\beta}_n(\kappa - \mathbb{E}X) \tag{24}$$

for all $n \in \mathbb{N}_0$, where coefficients $\hat{\alpha}_n^{(i)}$ and $\hat{\beta}_n$ are defined in Theorem 7.

Proof. In a similar way as is Lemma 6, the proof follows by induction. Proposition is obvious for $n = 0, \ldots, \kappa-2-l$ and follows by (19) for $n = \kappa-1-l$. Let $n \geqslant \kappa - l$. Then, from (20) we get:

$$\varphi(u+\kappa-l) = \frac{1}{h_l}\left(\varphi(u) - \sum_{i=1}^{u+\kappa-l-1} h_{u+\kappa-i}\varphi(i)\right).$$

Substituting $u = n - \kappa + l + 1$ into the last expression above and assuming induction hypothesis we obtain:

$$\varphi(n+1) = \frac{1}{h_l}\left(\varphi(n-\kappa+l+1) - \sum_{m=1}^{n} h_{n+l+1-m}\varphi(m)\right)$$

$$= \frac{1}{h_l}\left(\sum_{i=0}^{\kappa-2-l}\hat{\alpha}_{n+1-\kappa+l}^{(i)}\varphi(i) + \hat{\beta}_{n+1-\kappa+l}(\kappa - \mathbb{E}X) - \sum_{m=1}^{n} h_{n+l+1-m}\left(\sum_{i=0}^{\kappa-2-l}\hat{\alpha}_m^{(i)}\varphi(i) + \hat{\beta}_m(\kappa - \mathbb{E}X)\right)\right)$$
$$= \frac{1}{h_l}\left(\sum_{i=0}^{\kappa-2-l}\varphi(i)\left(\hat{\alpha}_{n+1-\kappa+l}^{(i)} - \sum_{m=1}^{n} h_{n+l+1-m}\hat{\alpha}_m^{(i)}\right) + (\kappa - \mathbb{E}X)\left(\hat{\beta}_{n+1-\kappa+l} - \sum_{m=1}^{n} h_{n+l+1-m}\hat{\beta}_m\right)\right)$$
$$= \sum_{i=0}^{\kappa-2-l}\hat{\alpha}_{n+1}^{(i)}\varphi(i) + \hat{\beta}_{n+1}(\kappa - \mathbb{E}X).$$

Hence, the equality (24) holds, and the lemma is proved. □

Proof of Theorems 7 and 8. follow from Lemma 7 and relations (19), (20). □

8. Discussion

The development of collective risk models is closely related to a random walk (r.w.), which is understood as a sum $\sum_{i=1}^{t} X_i$, where X_i are i.i.d. and $t \in \mathbb{N}$. The analogous description of ruin (or survival) probability is that a certain version of r.w. hits (or does not) some threshold for at least one (or none) $t \in \mathbb{N}$. As mentioned, our study of a certain version of r.w. in this work are both pure and insurance mathematics shifted. On the other hand, the range of r.w. applications is wide. Not repeating possible r.w. applications, mentioned in Introduction, we could add an example of some ecosystem where a certain amount individuals live, reproduce, and die with some level of randomness.

The split between finite and ultimate t is crucial. For example, for the model being investigated it holds that:

$$W(t) = u + \kappa t - \sum_{i=1}^{t} X_i = u + t - \sum_{i=1}^{t} X_i',$$

where $X_i \overset{d}{=} X_i' + \kappa - 1$. The last change of r.v. can be well utilized in view of that what is known for discrete time risk model survival or ruin probability calculation, see Section 5 in [15]. However, for finite time only.

Our obtained results apparently are similar to previously known non-homogeneous risk models, see [16,22,27–30], where certain convolutions of random variables occur and initial values for recurrent formulas are needed. Namely, convolutions of distinct r.v.s generating some discrete time non-homogeneous risk model is the reason for not allowing to easily express ultimate time survival probability. Furthermore, most likely it will spin around ultimate time survival probability of non-homogeneous risk models incorporating the generalized premium rate studied in this work. In addition, properties of determinants, defined by a certain "long memory" recurrent sequences, at some level, forms a new research branch.

9. Materials and Methods

Theoretical statements of this paper are obtained mainly using the strong law of large numbers, total probability and other technics from probability theory, mathematical induction, and elementary rearrangements. The computational part was carried out by program Mathematica and codes, on demand, are available from the authors. Our chosen claim distributions and other model inputs, used in numerical examples section, are purely theoretically driven.

10. Conclusions

In this work, we derived an exact formulas of survival probability calculation for the homogeneous discrete time risk model with generalized premium rate, which can be any natural number. In other words, it was proved and demonstrated via examples statements for likelihood that a certain random amount would never hit some threshold. The main focus, of course, was the ultimate time. This work supplements the research being done in the resent years by the same authors and others mentioned in the list of references. Our derived statements are mainly dependent on random variable and premium income rate κ generating the model. Of course, from a practical point of view, model inputs must be aligned with a situation where the model is being applied. A range of possible applications seems to be wide—just anywhere of what is expressible by increasing amount, decreasing amount, and level of randomness.

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