applied sciences

MDPI

Article

Non-Linear Creep-Relaxation Constitutive Damage Model for Aging Concrete

Bernardo T. Terán-Torres [1], Jamshid Mohammadi [2], Sudhakar E. Nair [3], José M. Mendoza-Rangel [1], Ismael Flores-Vivian [1] and César A. Juárez-Alvarado [1,*]

[1] Universidad Autónoma de Nuevo León, Facultad de Ingeniería Civil, Av. Universidad s/n, Cd. Universitaria, San Nicolás de los Garza 66455, NL, Mexico; bernardo.terantrr@uanl.edu.mx (B.T.T.-T.); jose.mendozarn@uanl.edu.mx (J.M.M.-R.); ismael.floresvvn@uanl.edu.mx (I.F.-V.)
[2] Department of Civil, Architectural and Environmental Engineering, Illinois Institute of Technology, 10 West 35th Street, Chicago, IL 60616, USA; mohammadi@iit.edu
[3] Department of Mechanical, Materials, and Aerospace Engineering, Illinois Institute of Technology, 10 West 35th Street, Chicago, IL 60616, USA; nair@iit.edu
* Correspondence: cesar.juarezal@uanl.edu.mx

Featured Application: Constitutive model for non-linear finite element analysis of concrete structures subjected to cyclic loading and sustained loading.

Abstract: A thermodynamic constitutive damage model for plain concrete, and other quasi-brittle aging materials, under creep relaxation is developed. The model accounts for the anisotropic damage induced through a second-order tensor damage variable. The aging viscoelasticity of the material is considered through the theory of solidification for aging solidifying materials. The material is considered a viscoelastic-damageable material. The Helmholtz free energy, utilized in the formulation, is treated based on the representation theorem of coupled damage strain tensors and Volterra integral equations. The model can analyze time-dependent damage (tertiary creep) under constant loading and can account for damage due to cyclic creep. Theoretical case studies are considered to illustrate the applicability of the model. The determination of the functions and constants, representing the material behavior, as well as any experimental companion is proposed for further research.

Keywords: concrete relaxation creep; constitutive damage model; aging viscoelasticity; tertiary creep

Citation: Terán-Torres, B.T.; Mohammadi, J.; Nair, S.E.; Mendoza-Rangel, J.M.; Flores-Vivian, I.; Juárez-Alvarado, C.A. Non-Linear Creep-Relaxation Constitutive Damage Model for Aging Concrete. *Appl. Sci.* **2024**, *14*, 4270. https://doi.org/10.3390/app14104270

Academic Editors: Angelo Luongo and Simona Di Nino

Received: 16 April 2024
Revised: 11 May 2024
Accepted: 13 May 2024
Published: 17 May 2024

1. Introduction

Concrete is one of the most important construction materials in use all over the world. It is an attractive material due to its versatility, easy fabrication, and low cost. Accordingly, concrete has been used in the construction of different types of structures, such as bridges, buildings, and dams, among others. However, due to its composite nature and quasi-brittle behavior, its mechanical behavior is difficult to predict under different loading conditions [1].

Particularly, in the case of bridges, which are subjected to a combined cyclic and sustained loading, recent studies [2,3] have shown that the current state of the art on models that account for purely sustained loading (creep) underestimate the long-term deflection, especially in long-span bridges. This underestimation has been attributed to the combined effect of sustained cyclic loading, known as cyclic creep. To account for such a phenomenon, a theory of cyclic creep based on linear elastic fracture mechanics was developed in [4]. It was found that cyclic creep has a negative impact on the long-term deflection of medium-span bridges (bridges with spans between 40 and 80 m). However, the theory has not been developed under the assumptions of the thermodynamic framework; moreover, concrete is assumed as a linear elastic material, which is an inaccurate representation of material behavior.

As an alternative to linear elastic fracture mechanics, Continuum Damage Mechanics (CDM) theory has been shown to have potential for the characterization of concrete mechanical behavior. Numerous studies in CDM theory for concrete under diverse conditions have been developed. For instance, damage models for brittle materials under monotonic, quasi-static, cyclic, and dynamic loading behavior were developed in [5–7]. In [8], a model based on mixture theory (separating the mortar and aggregate) and CDM was developed; the unilateral effect of cracks is primarily considered in this model. Some experiments and models related to the unilateral effect of cracks in concrete were conducted in [9,10]. CDM-bounding surface models for concrete under cyclic and monotonic loading were developed in [11,12]. Damage models coupled with plasticity were proposed in [13–23]; some of the features of these models were slip-type strain, multiaxial cyclic and monotonic loading behavior, isotropic and anisotropic damage, and the inclusion of damage tensor variables to account for compressive, tensile, and shear damage. Chaboche et al. [24] reformulated the theory of CDM for brittle materials by treating them as elastic damageable materials. Following this line of thought, a model for an elastic–brittle material with anisotropic damage was developed [25]; in this model, a representation theorem was first introduced to define the thermodynamic potential. The relationship between fracture and damage was studied in [26,27]. Damage models for concrete under hysteretic and fatigue behavior were developed in [28–31]. A damage model based on CDM was presented in [32]; in this model, the effects of initial damage are treated. Subsequently, a method to correctly determine the damage functions in quasi-brittle materials was obtained in [33]. The separation of hydrostatic and deviatoric components of damage were treated and discussed in [34]. In [35], a model for cyclic loading based on an isotropic damage variable was developed; stiffness recovery, inelastic strains, and frictional sliding were some of its features. Reviews of CDM and plasticity applied to concrete can be found in [36–38], and more recently in [39], where multiscale modeling, phase-field modeling, rate-dependent models, and fatigue were discussed.

One of the main challenges with the creep characterization of concrete is the change in the mechanical viscoelastic properties as concrete ages—a concept known as aging. Bažant was one of the first to recognize the aging of concrete and developed some early models [40,41]. Afterward, concrete aging was assumed to be related to the process of solidification of hydrated matter (cement gel) [42], and through this assumption, a theory for aging creep on concrete was formulated, i.e., the theory of solidification [42–45]. Unfortunately, the long-term estimations, obtained through [42], showed divergence in their graphical representation. This divergence was attributed to the utilization of the mean distance between capillary pores used in the creep function. Hence, the use of a different rheological model that considers a viscous flow strain component in the total strain was proposed in [43]. This viscous flow component in the strain is due to the migration of calcium silicate hydrate (CSH) particles inside the matrix [46]. This migration is the product of the sustained loading or drying of concrete. The solidification theory in [43,44] provides the theoretical framework for the derivation of the ACI B3 model [47]. The theory of solidification proposed in [43,44] accounted for the non-linear regime, i.e., for stresses larger than half the strength of concrete, where irreversible viscoplastic strain prevails. However, the non-linear theory was based only on curve fitting instead of a solid theoretical framework. The thermoviscoelasticity of the solidification theory was presented in [45]. A nonlinear model based on CDM and Bažant's theory of solidification was developed in [48]; in the model, an isotropic damage variable and a rate-type constitutive equation were used. One of the shortcomings was the function utilized for accounting damage in the viscoelastic part of the strain at high levels of stress based on empirical observations. Subsequently, a constitutive relation for brittle materials (concrete and masonry) based on rheological models and CDM was developed in [49]; the aging of concrete was absent in this constitutive relation. More recently, a new type of creep damage model has been developed in [50] to establish the relationship between creep parameters and time under different stresses. In a recent study [51], viscoelastic materials were mathematically modeled considering

molecular chain-network configurations and the temperature effects on the mechanical properties were characterized based on the equivalent relationship between temperature and frequency. Moreover, in [52], the mechanical behavior of a viscoelastic damper under monotonic and cyclic loading, focusing on the cyclic stability of mechanical properties, was explored.

In related research, and in the lead author's doctoral dissertation [53], a non-linear constitutive model for plain concrete under creep and relaxation was developed, and is presented in this paper. The model is based on thermodynamic constitutive theory, and it can account for creep under high levels of stress. In addition, the model has the potential to be utilized to account for cyclic creep and tertiary creep, since the damage is driven by a loading–unloading condition. The model is based on the Bažant solidification theory [43,45] and the CDM theory. The model is developed from the strain-based tensor formulation, meaning that the material functions that characterize the model are relaxation functions. Anisotropic damage is considered by using a second-order damage tensor variable, whose principal directions coincide with those of the strain tensor. The Helmholtz free energy is utilized as a measure of energy storage and thermodynamic potential. The Helmholtz free energy is defined based on the analogy of the representation theorem used by Murakami and Kamiya [25]. Finally, it has been assumed that the formulations used for brittle–elastic damageable materials [24] can also be applied to viscoelastic materials. Hence, the material is treated as viscoelastic damageable material. The model developed in the present article intends to provide a set of equations, with straightforward numerical implementation and thermodynamical consistency, that make it possible to obtain the behavior of bridges or other structures subjected to sustained and cyclic loading by means of finite element analysis. Although this research does not address the numerical implementation of the model, this can be attained through the numerical solution of the Volterra integral equations proposed. The determination of the functions and constants, representing the material behavior, as well as any experimental companion, is also proposed for further research. Section 2 addresses the theoretical formulation of the creep relaxation damage model. In Section 3, two case studies for the application of the model are presented. Conclusions on the theoretical model, along with the recommendations for further research, are listed in Section 4.

2. Theoretical Formulation

The model presented in this paper is developed under the consideration of the tensorial notation, where the indices $i, j, \ldots, N$ range from 1 to 3 indicate the three-dimensional space. In addition, Einstein's summation convention is implied when repeated indices appear.

Although the tensorial notation is utilized in most of this article, the direct notation (bolded characters) may also appear in some mathematical expressions, where the authors consider it suitable. Thus, bold symbols in this article represent vector and tensor fields.

Furthermore, the variables to be introduced in this formulation depend upon the position vector $\boldsymbol{x}$ and the current time variable t, unless otherwise specified. In addition, an auxiliary variable τ must be introduced to account for the loading time, and the loading duration is represented by the difference between these time variables, i.e., $(t - \tau)$.

2.1. Components of Strain and Thermodynamic Processes

2.1.1. Components of Strain

To formulate the Helmholtz free energy that accounts for the storage of energy in the plain concrete in the proposed formulation, let us first enlist the processes that take place in the concrete and the components of strain to be considered.

Firstly, to account for the aging process, a state variable $v(t)$ known as solidification [43,45] is considered in the formulation. Under high levels of stress (i.e., stress higher than $0.45f'c - 0.50f'c$ under compression), concrete develops microcracks that cause irreversible strain. Furthermore, under sustained load, the concrete undergoes time-dependent creep strain. Thus, under the former considerations and assuming the small strain regime,

where the state of the strain at a point is given by $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{x}, t)$, the state of the strain may be split into

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{ep} + \boldsymbol{\varepsilon}^{c} \tag{1}$$

where $\boldsymbol{\varepsilon}^{ep}$ accounts for the elastoplastic strain and $\boldsymbol{\varepsilon}^{c}$ represents the creep strain, both representing the strain for any level (low or high) of stress. It should be mentioned that $\boldsymbol{\varepsilon}^{ep}$ and $\boldsymbol{\varepsilon}^{c}$ can remain coupled if necessary (as it will be assumed for the case studies in Section 3), depending on the definition of the state of the strain, i.e., if the strain tensor contains the immediate and the temporal strain, respectively [54]. However, it is advantageous if the strain remains uncoupled, to measure them independently as well as their corresponding cracking (damage).

Secondly, since it has been well established in [43], creep strain for concrete has a short-term and a long-term component, where the latter is irreversible. Then, the creep strain may be further split into

$$\boldsymbol{\varepsilon}^{c} = \boldsymbol{\varepsilon}^{v} + \boldsymbol{\varepsilon}^{f} \tag{2}$$

where $\boldsymbol{\varepsilon}^{v}$ is the strain related to the short-term creep and $\boldsymbol{\varepsilon}^{f}$ is the viscous flow strain associated with long-term creep due to the migration of CSH particles. It shall also be noted that the short-term creep $\boldsymbol{\varepsilon}^{v}$ will represent the viscoelastic strain $\boldsymbol{\varepsilon}^{ve}$, viscoplastic strain $\boldsymbol{\varepsilon}^{vp}$, and visco-damage strain $\boldsymbol{\varepsilon}^{vd}$ components. In other words,

$$\boldsymbol{\varepsilon}^{v} = \boldsymbol{\varepsilon}^{v}\left(\boldsymbol{\varepsilon}^{ve}, \boldsymbol{\varepsilon}^{vp}, \boldsymbol{\varepsilon}^{vd}\right) \tag{3}$$

These latter strain components will remain coupled in the constitutive equation, as implied by Equation (3). The remaining subsections of this article will address the constitutive model related to the creep strain component $\boldsymbol{\varepsilon}^{c}$, and the remaining strains are proposed for further research. Figure 1 shows an approximated rheological chain model proposed for the complete set of strain components for the split state of the strain at any point, including current and future research models, where the springs represent conservative processes and the dashpots are dissipative.

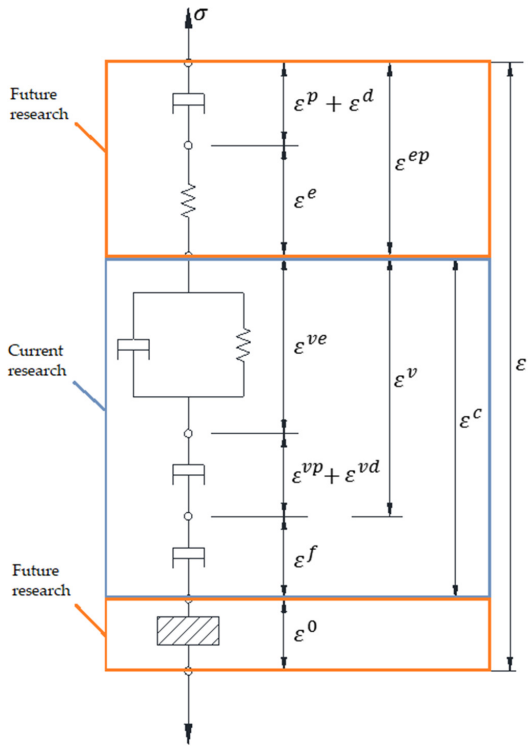

Figure 1. Strain components for the split state of strain at any point: ε^{0} shrinkage and thermal strain, $\boldsymbol{\varepsilon}^{c}$ creep strain, $\boldsymbol{\varepsilon}^{ep}$ elastoplastic strain, $\boldsymbol{\varepsilon}^{e}$ elastic strain, $\boldsymbol{\varepsilon}^{p}$ plastic strain, $\boldsymbol{\varepsilon}^{d}$ damage strain, $\boldsymbol{\varepsilon}^{v}$ short-term

creep strain, $\boldsymbol{\varepsilon}^{ve}$ viscoelastic strain, $\boldsymbol{\varepsilon}^{vp}$ viscoplastic strain, $\boldsymbol{\varepsilon}^{vd}$ visco-damage strain, and $\boldsymbol{\varepsilon}^{f}$ viscous flow strain (long-term creep).

2.1.2. Thermodynamic Processes

To formulate the constitutive equations that describe the behavior of plain concrete under cyclic creep, it is necessary to understand the different thermodynamic processes that take place in concrete upon loading. Energy due to these processes can be stored or dissipated, depending on whether the process is reversible or irreversible. In the case of concrete subjected to cyclic creep, and assuming that the material is also subjected to a uniform temperature and an isothermal process, these processes are the state of solidification, the state of strain and its magnitude, and hardening (which depends on both strain and solidification degree).

The state of solidification of the concrete is known to be associated with the heat of hydration, which is a result of the chemical reaction between water and cement, which leads to the solidification of the cement gel [42,43]. Although the concrete reaches the desired strength at 28 days, the process of solidification continues due to the new hydration of cement particles trapped inside the matrix or the solidification of new layers of cement gel inside the capillary pores. Therefore, this process aids the concrete to harden, by increasing the amount of volume of solidified matter in time [42,43].

Concrete is considered a viscoelastic aging material. Under the application of stress (strain), the material stores strain energy while the material is subjected to stress less than half of its strength. However, if the concrete is subjected to a stress higher than half its strength, the material develops microcracks that create an irreversible state of viscoplastic strain [43]. In CDM, these microcracks are known as damage. For the damage to develop, the material must store and dissipate energy [55]. Therefore, a process to account for damage growth must be taken into consideration. In addition, as the damage develops inside the material, strain hardening or softening occurs. Hence, a process known as damage strengthening must be considered.

Likewise, during casting, concrete behaves as a non-Newtonian fluid; and as previously mentioned, the solidification of concrete is a process that continues for some time. Therefore, it is reasonable to assume that particles of this non-Newtonian fluid remain trapped inside the matrix, such that, after sustained loading is applied, these particles start to flow. This is consistent with experimental data [46], where it has been shown that CSH particles, inside the matrix, migrate when the application of sustained loading occurs. These CSH particles do not return to their starting point once unloading takes place. This migration creates a permanent strain that cannot be recovered. Therefore, in [43], a viscous flow strain was considered to account for such microscopic behavior. Figure 2 summarizes the thermodynamic processes taking place in the concrete.

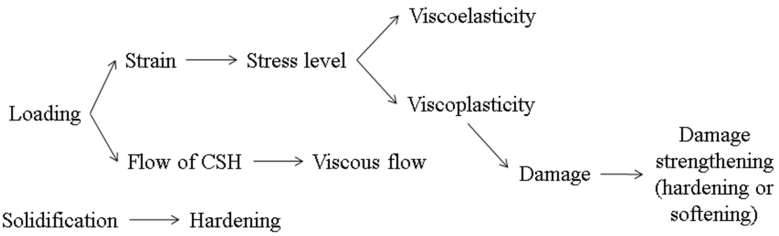

Figure 2. Thermodynamic processes that take place in the plain concrete during aging and loading.

2.2. Instantaneous Helmholtz Free Energy

2.2.1. Components of Helmholtz Free Energy

Based on the aforementioned discussions, the instantaneous Helmholtz free energy at any point can be written as follows:

$$\rho\Psi = \rho\Psi(\boldsymbol{x}, t) = \boldsymbol{\sigma} : \boldsymbol{\varepsilon}^{c} + \beta B + \boldsymbol{D} : \boldsymbol{Y} + vV \tag{4}$$

where ρ is the mass density, Ψ is the instantaneous Helmholtz free energy per unit mass, $\rho\Psi$ is the instantaneous Helmholtz free energy per unit volume, $\boldsymbol{\sigma} : \boldsymbol{\varepsilon}^c$ is strain energy associated with the creep strain component, βB is the energy associated with the damage strengthening (hardening and softening), $\boldsymbol{D} : \boldsymbol{Y}$ is the energy associated with the degree of cracking (damage), and vV is the energy associated with the solidification of the concrete. Hence, the state variables are as described below:

1. $\boldsymbol{\sigma}$ is the state of stress at the point;
2. $\boldsymbol{\varepsilon}^c$ is the state of creep strain;
3. β is the damage strengthening;
4. B is the thermodynamic conjugate of the damage strengthening;
5. $\boldsymbol{D}$ is the second-order damage tensor variable;
6. $\boldsymbol{Y}$ is the energy release rate associated to the damage;
7. v is the degree of solidification;
8. V is the thermodynamic conjugate of the degree of solidification.

It is important to indicate that all the state variables mentioned above, and their corresponding thermodynamic conjugates, not only depend on the position vector $\boldsymbol{x}$, but also on the time variable t. For instance, the state of stress is represented by $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{x}, t)$. The only exceptions are v and V, where position independence will be assumed, to simplify the formulation and to be consistent with the theory proposed in [45], i.e., $v = v(t)$ and $V = V(t)$. Also, from this point forward, the authors will also be referring to the instantaneous Helmholtz free energy as thermodynamic potential; and its position and time dependency will be implied but not stated in further equations, i.e., $\rho\Psi(\boldsymbol{x}, t) = \rho\Psi$, for simplification.

As stated above, the creep strain is split into two components, i.e., Equation (2). Hence, by substituting into Equation (4),

$$\rho\Psi = \boldsymbol{\sigma} : \boldsymbol{\varepsilon}^v + \boldsymbol{\sigma} : \boldsymbol{\varepsilon}^f + \beta B + \boldsymbol{D} : \boldsymbol{Y} + vV \tag{5}$$

Since the authors are considering Helmholtz free energy as thermodynamic potential, the independent state variables in the thermodynamic potential are the state of strain $\boldsymbol{\varepsilon}^v$ and $\boldsymbol{\varepsilon}^f$, the damage variable $\boldsymbol{D}$, the damage strengthening variable β, and the solidification variable v. The remaining variables $\boldsymbol{\sigma}$, $\boldsymbol{Y}$, B, and V are the dependent state variables, or thermodynamic conjugates to be found from the thermodynamic potential, i.e.,

$$\rho\Psi = \rho\Psi\left(\boldsymbol{\varepsilon}^v, \boldsymbol{\varepsilon}^f, \beta, \boldsymbol{D}, v\right) \tag{6}$$

The damage is considered to develop as the state of strain reaches a certain magnitude, which indicates the dependence of damage and strain. Also, although the strength to sustain damage (i.e., damage strengthening) depends on the amount of damage itself in the material, for the time being, it is assumed that the strength is uncoupled from the damage in the thermodynamic potential. However, these two entities (i.e., strength and damage) will be coupled later in the formulation when a dissipation potential is established. Furthermore, the viscous flow strain, although associated with stress, can be assumed to be independent of damage, because it is a process that appears at any level of strain. Finally, it is stipulated that the volume of solidified matter of concrete hardens the material, thereby making the material less susceptible to deformation. Therefore, the change in volume must be coupled to the strain, with both viscous flow strain and short-term creep strain. Hence, Equation (6) becomes

$$\rho\Psi = \rho\Psi^{v\varepsilon} + \rho\Psi^f + \rho\Psi^D \tag{7}$$

where $\rho\Psi^{v\varepsilon} = \rho\Psi^{v\varepsilon}(\boldsymbol{\varepsilon}^v, \boldsymbol{D}, v)$ is the strain energy associated with the short-term creep strain, damage, and volume of solidification, $\rho\Psi^f = \rho\Psi^f\left(\boldsymbol{\varepsilon}^f, v\right)$ is the strain energy associated with the viscous flow strain (long-term creep strain) and volume of solidification, and $\rho\Psi^D = \rho\Psi^D(\beta)$ is the energy associated with the damage strengthening. Equation (7) provides the components of the thermodynamic potential to be determined in this formulation.

2.2.2. Helmholtz Free Energy for a Short-Term Linear Viscoelastic Aging Material

Based on [45], the first term of the Helmholtz free energy stated in Equation (7), $\rho\Psi^{v\varepsilon}$, is obtained. The proof given in [45] addresses the thermodynamic potential for a unidimensional viscoelastic material. Nevertheless, the herein formulation leads to a further generalization of the equations, by extending the concepts to a three-dimensional case, assuming isotropy in the material, and the non-linear behavior by considering the development of the damage due to higher levels of stress. Accordingly, the material is assumed to be viscoelastic damageable material. In other words, the material is a viscoelastic recoverable material until the point where damage due to microcracking begins to develop, similar to that in [24].

Hence, assuming one solidification process to be sufficient for concrete [45], the thermodynamic potential for short-term viscoelastic aging material is obtained, which is expressed in its convolution Stieltjes integral form, i.e.,

$$\rho\Psi^{v\varepsilon} = \rho\int_{\theta=0}^{t}\psi(t,\theta)dv(\theta) = \frac{1}{2}\int_{\theta=0}^{t}\int_{\tau'=\theta}^{t}\int_{\tau=\theta}^{t}G_{ijkl}(2t-\tau-\tau')d\boldsymbol{\varepsilon}_{ij}^{v}(\tau)d\boldsymbol{\varepsilon}_{kl}^{v}(\tau')dv(\theta) \quad (8)$$

where G_{ijkl} is the age-independent relaxation tensor; θ is the time variable from which the time is zero, meaning that no-aging changes in the mechanical properties have occurred, up to the current time t; and τ' is an auxiliary variable that represents the loading time, similarly as τ.

If the most general form of an isotropic fourth-order tensor is considered,

$$G_{ijkl} = \frac{1}{3}(G_2 - G_1)\delta_{ij}\delta_{kl} + \frac{1}{2}G_1(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \quad (9)$$

where G_1 and G_2 are isotropic age-independent relaxation functions related to the deviatoric and dilatational strain, respectively, and δ_{ij} is the Kronecker delta. Furthermore, if the strain is split into dilatational and deviatoric, i.e.,

$$\boldsymbol{\varepsilon}_{ij}^{v} = \frac{1}{3}\boldsymbol{\varepsilon}_{kk}^{v}\delta_{ij} + e_{ij}^{v} \quad (10)$$

where the first term in Equation (10) represents the dilatational part of strain, and the second is the deviatoric part. Then, by the substitution Equation (9) into Equation (8), performing the changes in limit and order of integration represented in Figure 3, from Figure 3a–e, to discard the integration with respect to $v(\theta)$, and by substituting Equation (10) and considering the symmetry of the strain tensor, one finds

$$\rho\Psi^{v\varepsilon} = \int_{\tau'=0}^{t}\int_{\tau=\tau'}^{t}\frac{1}{3}\hat{G}_2(2t-\tau,\tau')d\boldsymbol{\varepsilon}_{ii}^{v}(\tau)d\boldsymbol{\varepsilon}_{jj}^{v}(\tau') + \int_{\tau'=0}^{t}\int_{\tau=\tau'}^{t}\hat{G}_1(2t-\tau,\tau')de_{ij}^{v}(\tau)de_{ij}^{v}(\tau') \quad (11)$$

where $\hat{G}_1$ and $\hat{G}_2$ are isotropic age-dependent (aging) relaxation functions related to the deviatoric and dilatational strain, given by Equation (12). It is important to note that, from this point forward, the circumflex accent $\hat{}$ will represent aging functions that depend on the variable $v(t)$. Hence, $\hat{G}_1$ and $\hat{G}_2$ are written as

$$\begin{aligned}\hat{G}_1(2t-\tau,\tau') &= v(\tau')G_1(2t-\tau-\tau')\\ \hat{G}_2(2t-\tau,\tau') &= v(\tau')G_2(2t-\tau-\tau')\end{aligned} \quad (12)$$

Equation (11) gives the energy stored in the isotropic linear viscoelastic aging material due to the application of a three-dimensional state of strain with dilatational and deviatoric components for a non-symmetric temporal region of integration. This equation can be rewritten in a more familiar form if the isotropic relaxation aging functions, $\hat{G}_1$ and $\hat{G}_2$, which are related to Lamé relaxation aging functions [54], i.e.,

$$\begin{aligned}\hat{\lambda}(2t-\tau,\tau') &= \tfrac{1}{3}\left[\hat{G}_2(2t-\tau,\tau') - \hat{G}_1(2t-\tau,\tau')\right]\\ \hat{\mu}(2t-\tau,\tau') &= \tfrac{1}{2}\hat{G}_1(2t-\tau,\tau')\end{aligned} \quad (13)$$

with Equation (13) into Equation (9) and following the previous steps, Equation (11) becomes the Helmholtz free energy for an isotropic linear viscoelastic aging material defined for a non-symmetric temporal region of integration, Equation (14), which provides the amount of energy stored in the material due to a state of short-term creep strain before damage has occurred, i.e.,

$$\rho\Psi^{v\varepsilon} = \int_{\tau'=0}^{t}\int_{\tau=\tau'}^{t} \hat{\lambda}(2t-\tau,\tau')d\boldsymbol{\varepsilon}_{ii}^{v}(\tau)d\boldsymbol{\varepsilon}_{jj}^{v}(\tau') + 2\int_{\tau'=0}^{t}\int_{\tau=\tau'}^{t} \hat{\mu}(2t-\tau,\tau')d\boldsymbol{\varepsilon}_{ij}^{v}(\tau)d\boldsymbol{\varepsilon}_{ij}^{v}(\tau') \tag{14}$$

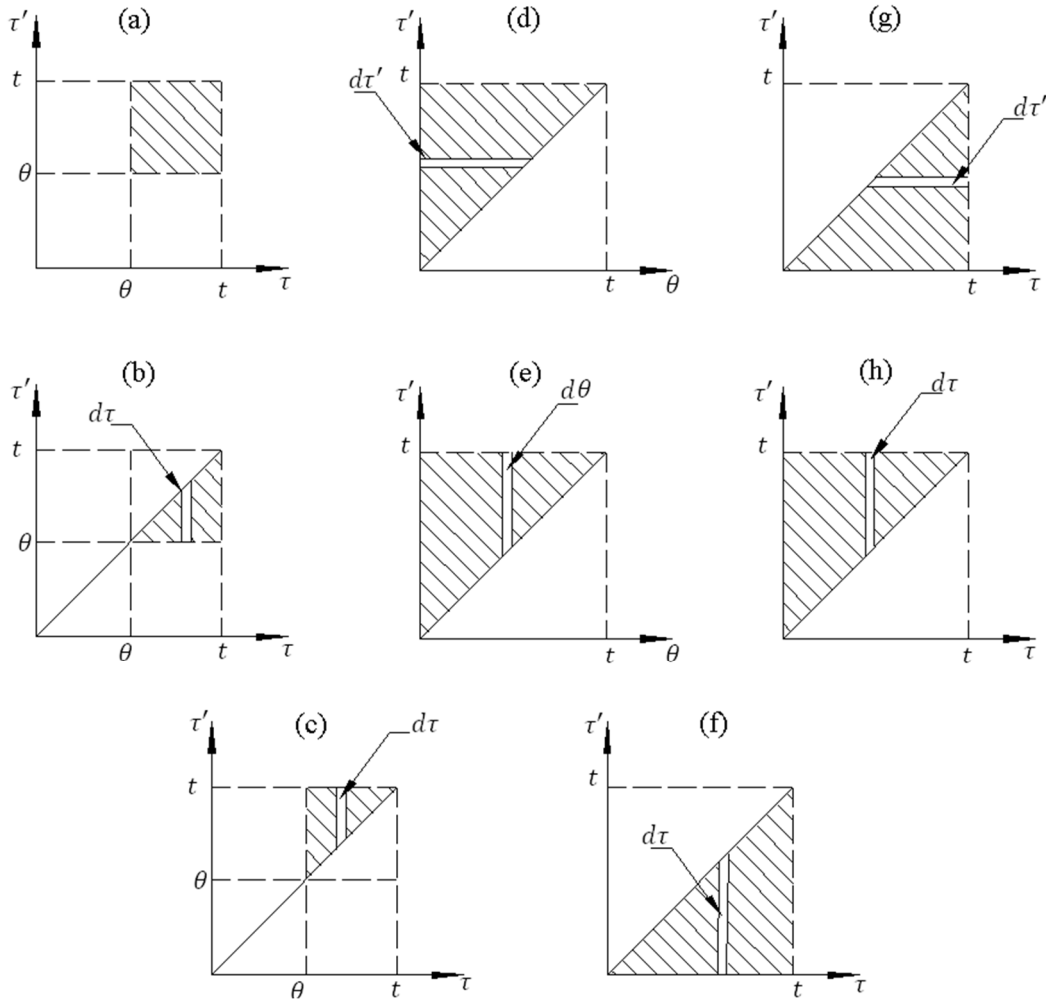

Figure 3. Changes in order and limits of integration for the instantaneous Helmholtz free energy of an isotropic linear viscoelastic aging material. (**a**) Symmetric integration from θ to t in $\tau-\tau'$ domain; (**b**) Integration from $\tau=\tau'$ to t with respect τ in $\tau-\tau'$ domain; (**c**) Integration from θ to $\tau=\tau'$ with respect τ in $\tau-\tau'$ domain; (**d**) Integration from $\tau'=\theta$ to t with respect τ' in $\theta-\tau'$ domain; (**e**) Integration from 0 to $\theta=\tau'$ with respect θ in $\theta-\tau'$ domain; (**f**) Integration from $\tau=\tau'$ to t with respect τ in $\tau-\tau'$ domain; (**g**) Integration from 0 to t with respect τ' in $\tau-\tau'$ domain; (**h**) Integration from 0 to $\tau=\tau'$ with respect τ in $\tau-\tau'$ domain.

The symmetry in the temporal region of integration can be attained if the change in limits and order of integration represented in Figure 3, from Figure 3f–h, are carried out in Equation (14), and then, by expressing the equation as a semi sum of its terms,

$$\begin{aligned}\rho\Psi^{v\varepsilon} = \tfrac{1}{2}\int_{\tau'=0}^{t}\Big[& \int_{\tau=\tau'}^{t}\hat{\lambda}(2t-\tau,\tau')d\boldsymbol{\varepsilon}_{ii}^{v}(\tau) + \int_{\tau=0}^{\tau'}\hat{\lambda}(2t-\tau',\tau)d\boldsymbol{\varepsilon}_{ii}^{v}(\tau)\Big]d\boldsymbol{\varepsilon}_{jj}^{v}(\tau') \\ & + \int_{\tau'=0}^{t}\Big[\int_{\tau=\tau'}^{t}\hat{\mu}(2t-\tau,\tau')d\boldsymbol{\varepsilon}_{ij}^{v}(\tau) + \int_{\tau=0}^{\tau'}\hat{\mu}(2t-\tau',\tau)d\boldsymbol{\varepsilon}_{ij}^{v}(\tau)\Big]d\boldsymbol{\varepsilon}_{ij}^{v}(\tau')\end{aligned} \tag{15}$$

As a final step, we couple Equation (15) with the Lamé relaxation aging functions. Furthermore, we use a Heaviside temporal function, $H_t(*)$, which is defined as 1 when the term inside the parenthesis, i.e., $(*)$, is positive, $\frac{1}{2}$ when $(*)$ is zero, and zero when $(*)$ is negative. It is important to mention that, from this point forward, the tilde $\widetilde{*}$ will represent relaxation aging functions with a temporal Heaviside function defined inside. Hence, the Lamé relaxation aging functions are rewritten as

$$\begin{aligned}\widetilde{\lambda}(t,\tau,\tau') &= H_t(\tau-\tau')\hat{\lambda}(2t-\tau,\tau') + H_t(\tau'-\tau)\hat{\lambda}(2t-\tau',\tau)\\ \widetilde{\mu}(t,\tau,\tau') &= H_t(\tau-\tau')\hat{\mu}(2t-\tau,\tau') + H_t(\tau'-\tau)\hat{\mu}(2t-\tau',\tau)\end{aligned} \tag{16}$$

and thus, the symmetry in the integration region is attained in Equation (17). Figure 4 represents the regions of integration for both $\widetilde{\lambda}$ and $\widetilde{\mu}$.

$$\rho\Psi^{v\varepsilon} = \frac{1}{2}\int_{\tau'=0}^{t}\int_{\tau=0}^{t}\widetilde{\lambda}(t,\tau,\tau')d\varepsilon_{ii}^{v}(\tau)d\varepsilon_{jj}^{v}(\tau') + \int_{\tau'=0}^{t}\int_{\tau=0}^{t}\widetilde{\mu}(t,\tau,\tau')d\varepsilon_{ij}^{v}(\tau)d\varepsilon_{ij}^{v}(\tau') \tag{17}$$

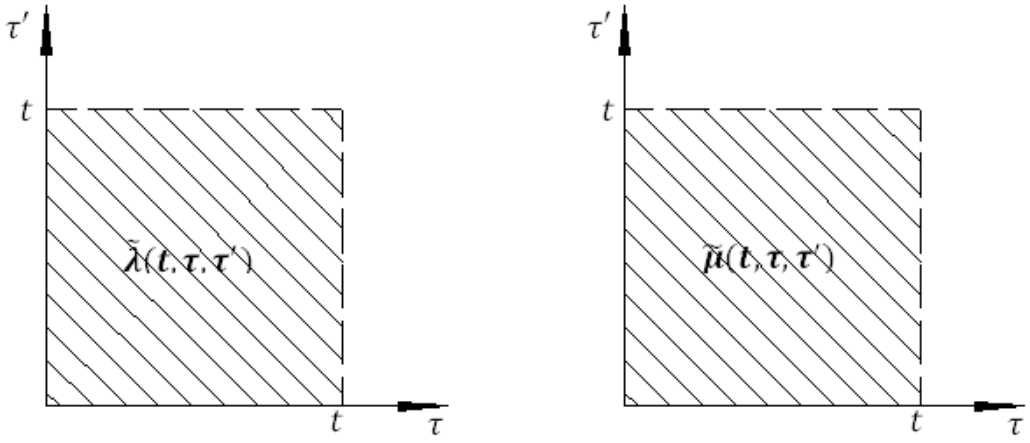

Figure 4. Symmetric region of integration for the Lamé aging relaxation functions defined for thermodynamic potential.

Equation (17) is the thermodynamic potential from which a constitutive relation between stress and strain can be obtained for an isotropic linear viscoelastic aging material. Equation (17) can be further simplified to express the stress–strain relation for a uniaxial material, as later discussed in Section 3.2 of this paper. In addition, and similar to Equations (11) and (14), the thermodynamic potential in Equation (17) provides the energy stored in the material for a symmetric temporal region of integration, from which the integration can be attained in a straightforward manner. Equation (17) is analogous to the Helmholtz free energy for a linear elastic isotropic material, Equation (18), where $tr(*)$ is the trace of the tensor inside the parenthesis, and λ and μ are Lamé constants.

$$\rho\psi = \frac{1}{2}\lambda[tr(\boldsymbol{\varepsilon})]^2 + \mu\left[tr\left(\boldsymbol{\varepsilon}^2\right)\right] = \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij} \tag{18}$$

Equation (18) is represented in two forms. The first form is represented by a quadratic scalar-valued tensor function in its general form based on the invariants of the strain tensor field [55], i.e., the traces $tr(\boldsymbol{\varepsilon})$, $tr\left(\boldsymbol{\varepsilon}^2\right)$, and $tr\left(\boldsymbol{\varepsilon}^3\right)$, where only quadratic terms are considered to obtain a linear relation between stress–strain. The second form is the component form, or tensorial notation, from which it is easier to notice the similarity between Equations (17) and (18). In addition, from both forms of Equation (18), it is possible to obtain the stress–strain relation for a linear elastic isotropic material, by the differentiation of Equation (18) with respect to the strain tensor components $\boldsymbol{\varepsilon}_{ij}$.

2.2.3. Helmholtz Free Energy for a Short-Term Non-Linear Viscoelastic Aging Material with Unilateral Effect of Damage

As stated before, a symmetric second-order damage tensor variable is considered, written in terms of its spectral decomposition, i.e.,

$$\boldsymbol{D} = \sum_{i=1}^{3} \boldsymbol{D}_i \boldsymbol{n}^{(i)D} \otimes \boldsymbol{n}^{(i)D} \tag{19}$$

where $\boldsymbol{D}_i$ are the eigenvalues of the damage tensor, and $\boldsymbol{n}^{(i)D}$ are the eigenvectors of the damage tensor. In addition, if the principal directions of the damage tensor are assumed to coincide with those of the strain tensor, i.e., $\boldsymbol{n}^{(i)D} = \boldsymbol{n}^{(i)\varepsilon}$, where $\boldsymbol{n}^{(i)\varepsilon}$ are the eigenvectors of the strain tensor. Hence, the thermodynamic potential presented in Equation (18) can be further expanded to express the Helmholtz free energy as a function of the ten general invariants of strain and damage [55]. For an elastic material, the ten invariants are $tr(\boldsymbol{\varepsilon})$, $tr(\boldsymbol{\varepsilon}^2)$, $tr(\boldsymbol{\varepsilon}^3)$, $tr(\boldsymbol{D})$, $tr(\boldsymbol{D}^2)$, $tr(\boldsymbol{D}^3)$, $tr(\boldsymbol{\varepsilon D})$, $tr(\boldsymbol{\varepsilon D}^2)$, $tr(\boldsymbol{\varepsilon}^2\boldsymbol{D})$, and $tr(\boldsymbol{\varepsilon}^2\boldsymbol{D}^2)$. Therefore, for a linear relation between strain and stress, only terms with quadratic strain are used; and those terms with lower or higher order than quadratic are neglected. In the same manner, it is assumed that the damage affects the strain–stress relation linearly. Thus, the quadratic and other higher-order terms for the damage variable are neglected. Therefore, in direct notation and component form, respectively,

$$\begin{aligned}\rho\psi = \rho\psi(\boldsymbol{\varepsilon}, \boldsymbol{D}) = \; & \tfrac{1}{2}\lambda[tr(\boldsymbol{\varepsilon})][tr(\boldsymbol{\varepsilon})] + \mu(\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}) + \chi_1[tr(\boldsymbol{D})][tr(\boldsymbol{\varepsilon})][tr(\boldsymbol{\varepsilon})] + \chi_2[tr(\boldsymbol{D})](\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}) + \chi_3[tr(\boldsymbol{\varepsilon})](\boldsymbol{\varepsilon} : \boldsymbol{D}) \\ & + \chi_4[\boldsymbol{\varepsilon} : (\boldsymbol{\varepsilon D})]\end{aligned} \tag{20}$$

$$\rho\psi = \frac{1}{2}\lambda\boldsymbol{\varepsilon}_{ii}\boldsymbol{\varepsilon}_{jj} + \mu\boldsymbol{\varepsilon}_{ij}\boldsymbol{\varepsilon}_{ij} + \chi_1\boldsymbol{D}_{ii}\boldsymbol{\varepsilon}_{jj}\boldsymbol{\varepsilon}_{kk} + \chi_2\boldsymbol{D}_{kk}\boldsymbol{\varepsilon}_{ij}\boldsymbol{\varepsilon}_{ij} + \chi_3\boldsymbol{\varepsilon}_{kk}\boldsymbol{\varepsilon}_{ij}\boldsymbol{D}_{ij} + \chi_4\boldsymbol{\varepsilon}_{ij}\boldsymbol{\varepsilon}_{ik}\boldsymbol{D}_{kj} \tag{21}$$

With Equations (17) and (21), the Helmholtz free energy can be expressed to account for the damage. If the rate of change in the damage variable (evolution equation) was assumed to be small in magnitude, the rate of change in the evolution of the damage would be even smaller, so that it could be neglected. Thus, the terms related to the rate of change in the damage variable in the energy equation can also be neglected, since they depend on the rate of evolution. Therefore, the Helmholtz free energy of a viscoelastic aging isotropic material coupled with anisotropic damage, Equation (22), can be written as

$$\begin{aligned}\rho\Psi^{v\varepsilon} = \tfrac{1}{2}\int_{\tau'=0}^{t} \; & \int_{\tau=0}^{t} \widetilde{\lambda}(t,\tau,\tau')d\boldsymbol{\varepsilon}^v_{ii}(\tau)d\boldsymbol{\varepsilon}^v_{jj}(\tau') + \int_{\tau'=0}^{t}\int_{\tau=0}^{t} \widetilde{\mu}(t,\tau,\tau')d\boldsymbol{\varepsilon}^v_{ij}(\tau)d\boldsymbol{\varepsilon}^v_{ij}(\tau') \\ & + \int_{\tau'=0}^{t}\int_{\tau=0}^{t} \widetilde{\chi}_1(t,\tau,\tau')\boldsymbol{D}_{ii}d\boldsymbol{\varepsilon}^v_{jj}(\tau)d\boldsymbol{\varepsilon}^v_{kk}(\tau') \\ & + \int_{\tau'=0}^{t}\int_{\tau=0}^{t} \widetilde{\chi}_2(t,\tau,\tau')\boldsymbol{D}_{kk}d\boldsymbol{\varepsilon}^v_{ij}(\tau)d\boldsymbol{\varepsilon}^v_{ij}(\tau') + \int_{\tau'=0}^{t}\int_{\tau=0}^{t} \widetilde{\chi}_3(t,\tau,\tau')\boldsymbol{D}_{ij}d\boldsymbol{\varepsilon}^v_{kk}(\tau)d\boldsymbol{\varepsilon}^v_{ij}(\tau') \\ & + \int_{\tau'=0}^{t}\int_{\tau=0}^{t} \widetilde{\chi}_4(t,\tau,\tau')\boldsymbol{D}_{jk}d\boldsymbol{\varepsilon}^v_{ij}(\tau)d\boldsymbol{\varepsilon}^v_{ik}(\tau')\end{aligned} \tag{22}$$

where the relaxation material functions $\widetilde{\chi}_I(t,\tau,\tau')$ (with $I = 1,2,3,4$) must be determined from experimental data; and they are assumed to be dependent on the degree of solidification in the material, that is,

$$\widetilde{\chi}_I(t,\tau,\tau') = H_t(\tau-\tau')\hat{\chi}_I(2t-\tau,\tau') + H_t(\tau'-\tau)\hat{\chi}_I(2t-\tau',\tau) \tag{23}$$

where $\hat{\chi}_I$ is the aging relaxation function containing the solidification parameter, i.e.,

$$\hat{\chi}_I(2t-\tau,\tau') = v(\tau')\chi_I(2t-\tau-\tau') \tag{24}$$

It is well established in the literature that cracks in concrete grow depending on the sign of stress [6–39]. For instance, if the concrete is subjected to a tensile stress, microcracks grow perpendicular to the direction of the stress. Meanwhile, if it undergoes compressive stress (splitting stress), the microcracks grow parallel to the direction of the stress. In other words, in both cases, microcracks grow in the direction of the positive strain. Figure 5

shows how microcracks develop inside the material due to the different signs of stress. To account for this behavior, the positive part of the strain tensor must be assumed as [25]

$$\boldsymbol{\varepsilon}^{v+} = \mathbb{P}^{\varepsilon+} : \boldsymbol{\varepsilon}^{v} \tag{25}$$

where $\mathbb{P}^{\varepsilon+}$ is a fourth-order tensor known as the positive orthogonal projection of the strain [56], defined as

$$\mathbb{P}^{\varepsilon+} = \boldsymbol{Q}^{\varepsilon+} \otimes \boldsymbol{Q}^{\varepsilon+} \tag{26}$$

where $\boldsymbol{Q}^{\varepsilon+}$ is the positive strain rotation tensor given by the multiplication of the tensor product of the eigenvectors of the strain tensor $\boldsymbol{n}^{(i)\varepsilon}$ acting on itself, and a Heaviside function defined in the principal strain space, $H_{\varepsilon}(\boldsymbol{\varepsilon}_i^{v})$, where $\boldsymbol{\varepsilon}_i^{v}$ are the eigenvalues of the short-term creep strain tensor, i.e.,

$$\boldsymbol{Q}^{\varepsilon+} = \sum_{i=1}^{3} H_{\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}_i^{v}) \boldsymbol{n}^{(i)\varepsilon} \otimes \boldsymbol{n}^{(i)\varepsilon} \tag{27}$$

where $H_{\boldsymbol{\varepsilon}}$ is defined as 1 if $\boldsymbol{\varepsilon}_i^{v}$ is positive, and 0 when $\boldsymbol{\varepsilon}_i^{v}$ is either negative or zero. Thus, if the spectral decomposition of $\boldsymbol{\varepsilon}^{v}$ is considered, similar to Equation (19), it is easily proven that Equation (25) is

$$\boldsymbol{\varepsilon}^{v+} = \sum_{i=1}^{3} H_{\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}_i^{v}) \boldsymbol{\varepsilon}_i^{v} \boldsymbol{n}^{(i)\varepsilon} \otimes \boldsymbol{n}^{(i)\varepsilon} \tag{28}$$

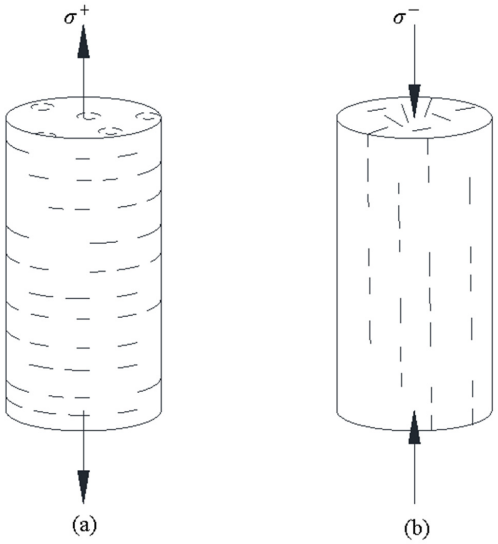

Figure 5. Microcracks in concrete specimens: (**a**) Specimen in tension; (**b**) Specimens in compression.

Therefore, upon replacing $\boldsymbol{\varepsilon}^{v}$ by $\boldsymbol{\varepsilon}^{v+}$ in the terms with $\boldsymbol{D}$ in Equation (22), the Helmholtz free energy of a viscoelastic aging isotropic material coupled with anisotropic damage with unilateral effect is found as Equation (29)

$$\begin{aligned}\rho\Psi^{v\varepsilon} = \tfrac{1}{2}\int_{\tau'=0}^{t} & \int_{\tau=0}^{t} \widetilde{\lambda}(t,\tau,\tau') d\varepsilon_{ii}^{v}(\tau) d\varepsilon_{jj}^{v}(\tau') + \int_{\tau'=0}^{t}\int_{\tau=0}^{t} \widetilde{\mu}(t,\tau,\tau') d\varepsilon_{ij}^{v}(\tau) d\varepsilon_{ij}^{v}(\tau') \\ & + \int_{\tau'=0}^{t}\int_{\tau=0}^{t} \widetilde{\chi}_1(t,\tau,\tau') D_{ii} d\varepsilon_{jj}^{v+}(\tau) d\varepsilon_{kk}^{v+}(\tau') \\ & + \int_{\tau'=0}^{t}\int_{\tau=0}^{t} \widetilde{\chi}_2(t,\tau,\tau') D_{kk} d\varepsilon_{ij}^{v+}(\tau) d\varepsilon_{ij}^{v+}(\tau') + \int_{\tau'=0}^{t}\int_{\tau=0}^{t} \widetilde{\chi}_3(t,\tau,\tau') D_{ij} d\varepsilon_{kk}^{v+}(\tau) d\varepsilon_{ij}^{v+}(\tau') \\ & + \int_{\tau'=0}^{t}\int_{\tau=0}^{t} \widetilde{\chi}_4(t,\tau,\tau') D_{jk} d\varepsilon_{ij}^{v+}(\tau) d\varepsilon_{ik}^{v+}(\tau')\end{aligned} \tag{29}$$

where ε_{ij}^{v+} and ε_{ii}^{v+} are given by the double contraction of the positive orthogonal projection components $P_{ijkl}^{\varepsilon+}$ and the short-term creep strain components ε_{ij}^{v}, defined in Equation (25); hence,

$$\varepsilon_{ij}^{v+} = P_{ijkl}^{\varepsilon+} \varepsilon_{kl}^{v} \varepsilon_{ii}^{v+} = P_{iikl}^{\varepsilon+} \varepsilon_{kl}^{v} \tag{30}$$

Equation (29) provides the first term on the right-hand side of the thermodynamic potential expressed in Equation (7). Hence, it is important to determine the remaining terms in this equation, i.e., $\rho\Psi^f$ and $\rho\Psi^D$. It should be noted that the development followed to determine $\rho\Psi^{v\epsilon}$, Equation (29), has been difficult to handle. However, the remaining terms in Equation (7) are determined in a straightforward manner.

2.2.4. Helmholtz Free Energy for Viscous Flow Strain and Damage Strengthening

For the second term in the right-hand side of Equation (7), let us split the stress into hydrostatic and deviatoric parts, i.e., $\boldsymbol{\sigma} = -p\boldsymbol{I} + \boldsymbol{s}$, where p is the hydrostatic pressure given by $p = -\frac{1}{3}tr(\boldsymbol{\sigma})$, $\boldsymbol{I}$ is the identity tensor, and $\boldsymbol{s}$ is the deviatoric stress tensor. By taking the hydrostatic and deviatoric parts of the stress, and substituting them into the second term in the right-hand side of Equation (5), i.e., $\boldsymbol{\sigma} : \boldsymbol{\varepsilon}^f$, and assuming a linear relationship, we obtain second term in the right-hand side of Equation (7),

$$\rho\Psi^f = -\frac{1}{2}p\,(\boldsymbol{I} : \boldsymbol{\varepsilon}^f) + \frac{1}{2}\boldsymbol{s} : \boldsymbol{\varepsilon}^f \tag{31}$$

From this equation, it is easily recognized that the term $\boldsymbol{s} : \boldsymbol{\varepsilon}^f$ is zero since a viscous flow is a process entirely dissipative when subjected to shear stress [57]. Furthermore, it is also easily noted that $\boldsymbol{I} : \boldsymbol{\varepsilon}^f = tr\,(\boldsymbol{\varepsilon}^f)$. Subsequently, from an analogy with linear elasticity,

$$\rho\Psi^f = -\frac{1}{2}p\,(\boldsymbol{I} : \boldsymbol{\varepsilon}^f) = \frac{1}{2}\hat{k}_f\,[tr\,(\boldsymbol{\varepsilon}^f)]^2 \tag{32}$$

where the relationship $p = -\hat{k}_f tr\,(\boldsymbol{\varepsilon}^f)$ was assumed, in which $\hat{k}_f = \hat{k}_f(t)$ is the long-term creep bulk aging function, related to the long-term creep Lamé aging functions $\hat{\lambda}_f$ and $\hat{\mu}_f$ through $\hat{k}_f = \hat{\lambda}_f + \frac{2}{3}\hat{\mu}_f$. Aging functions are considered since they may depend on the variable $v(t)$ [43,45]. In addition, Equation (32) can also be written in component form,

$$\rho\Psi^f = \frac{1}{2}\hat{k}_f\varepsilon^f_{ii}\varepsilon^f_{jj} \tag{33}$$

Equation (33) is the Helmholtz free energy that accounts for the energy stored in the material due to the long-term creep state of the strain. It is relevant to note that the long-term creep Lamé aging functions $\hat{\lambda}_f$ and $\hat{\mu}_f$, related to $\hat{k}_f$, do not necessarily coincide with those relaxation functions defined in Equation (16).

Finally, the third term in Equation (7) is assumed to be given by

$$\rho\Psi^D = \frac{1}{2}\hat{k}_D\beta^2 \tag{34}$$

defined similarly as in [25], and where, $\hat{k}_D = \hat{k}_D(t)$ is an experimental aging hardening function associated with hardening and the age of the quasi-brittle material.

2.2.5. Thermodynamic Potential

We now combine Equations (7), (29), (33), and (34), to obtain the thermodynamic potential, i.e.,

$$\begin{aligned}\rho\Psi = \tfrac{1}{2}\int_{\tau'=0}^{t}\int_{\tau=0}^{t} \;& \widetilde{\lambda}(t,\tau,\tau')d\varepsilon^v_{ii}(\tau)d\varepsilon^v_{jj}(\tau') + \int_{\tau'=0}^{t}\int_{\tau=0}^{t}\widetilde{\mu}(t,\tau,\tau')d\varepsilon^v_{ij}(\tau)d\varepsilon^v_{ij}(\tau') \\ &+ \int_{\tau'=0}^{t}\int_{\tau=0}^{t}\widetilde{\chi}_1(t,\tau,\tau')D_{ii}d\varepsilon^{v+}_{jj}(\tau)d\varepsilon^{v+}_{kk}(\tau') \\ &+ \int_{\tau'=0}^{t}\int_{\tau=0}^{t}\widetilde{\chi}_2(t,\tau,\tau')D_{kk}d\varepsilon^{v+}_{ij}(\tau)d\varepsilon^{v+}_{ij}(\tau') + \int_{\tau'=0}^{t}\int_{\tau=0}^{t}\widetilde{\chi}_3(t,\tau,\tau')D_{ij}d\varepsilon^{v+}_{kk}(\tau)d\varepsilon^{v+}_{ij}(\tau') \\ &+ \int_{\tau'=0}^{t}\int_{\tau=0}^{t}\widetilde{\chi}_4(t,\tau,\tau')D_{jk}d\varepsilon^{v+}_{ij}(\tau)d\varepsilon^{v+}_{ik}(\tau') + \tfrac{1}{2}\hat{k}_f\varepsilon^f_{ii}\varepsilon^f_{jj} + \tfrac{1}{2}\hat{k}_D\beta^2\end{aligned} \tag{35}$$

Equation (35) is the thermodynamic potential from which all the constitutive and evolution equations of the dissipative processes are determined. Equation (35) also provides

the complete form of the instantaneous Helmholtz free energy, which accounts for the energy storage in the material.

2.3. Constitutive Equations

2.3.1. Creep Strain–Stress Relation

The strain stress relation is given by the sum of the partial derivative of the thermodynamic potential with respect to the creep strain components, i.e., $\sigma_{ij} = \rho\frac{\partial\Psi}{\partial\varepsilon^{v}_{ij}} + \rho\frac{\partial\Psi}{\partial\varepsilon^{f}_{ij}}$. Then, by using Equation (35), the Leibniz rule, and changing τ and τ' equal to t, such that $H_t(\tau' - t) = 0$, when $\tau = t$, and $H_t(\tau - t) = 0$, when $\tau' = t$. Finally, by setting $\tau = \tau'$, treating the positive orthogonal projection $P^{\varepsilon+}_{ijkl}$ as constant, and assuming the symmetry of the damage tensor, one finds

$$\begin{aligned}\sigma_{ij} = \rho\frac{\partial\Psi}{\partial\varepsilon^{v}_{ij}} + \rho\frac{\partial\Psi}{\partial\varepsilon^{f}_{ij}} &= \int_{\tau=0}^{t}\hat{\lambda}(t,\tau)d\varepsilon^{v}_{kk}(\tau)\delta_{ij} + 2\int_{\tau=0}^{t}\hat{\mu}(t,\tau)d\varepsilon^{v}_{ij}(\tau) + 2\int_{\tau=0}^{t}\hat{\chi}_1(t,\tau)D_{kk}P^{\varepsilon+}_{mmij}d\varepsilon^{v+}_{ll}(\tau)\\ &+2\int_{\tau=0}^{t}\hat{\chi}_2(t,\tau)D_{mm}P^{\varepsilon+}_{klij}d\varepsilon^{v+}_{kl}(\tau) + \int_{\tau=0}^{t}\hat{\chi}_3(t,\tau)D_{kl}P^{\varepsilon+}_{mmij}d\varepsilon^{v+}_{kl}(\tau)\\ &+\int_{\tau=0}^{t}\hat{\chi}_3(t,\tau)D_{kl}P^{\varepsilon+}_{klij}d\varepsilon^{v+}_{mm}(\tau) + 2\int_{\tau=0}^{t}\hat{\chi}_4(t,\tau)D_{lm}P^{\varepsilon+}_{kmij}d\varepsilon^{v+}_{kl}(\tau) + \hat{k}_f\varepsilon^{f}_{kk}\delta_{ij}\end{aligned} \tag{36}$$

where the relaxation functions are dependent on the aging term $v(t)$ at loading time τ, i.e., $v(\tau)$, and the non-aging relaxation functions evaluated at loading duration $(t - \tau)$. In mathematical terms: $\hat{\lambda}(t,\tau) = v(\tau)\lambda(t-\tau)$, $\hat{\mu}(t,\tau) = v(\tau)\mu(t-\tau)$, and $\hat{\chi}_I(t,\tau) = v(\tau)\chi_I(t-\tau)$ (with $I = 1,2,3,4$).

2.3.2. Energy Released Rate

The energy release rate is given by the negative partial derivative of the thermodynamic potential with respect to the damage strain components, i.e., $Y_{ij} = -\rho\frac{\partial\Psi}{\partial D_{ij}}$. With Equation (35), the Leibniz rule, and by performing some mathematical operations, the amount of energy released due to the damage growth is obtained:

$$\begin{aligned}Y_{ij} = -\rho\frac{\partial\Psi}{\partial D_{ij}} = &-\int_{\tau'=0}^{t}\int_{\tau=0}^{t}\widetilde{\chi}_1(t,\tau,\tau')d\varepsilon^{v+}_{kk}(\tau)d\varepsilon^{v+}_{ll}(\tau')\delta_{ij} - \int_{\tau'=0}^{t}\int_{\tau=0}^{t}\widetilde{\chi}_2(t,\tau,\tau')d\varepsilon^{v+}_{kl}(\tau)d\varepsilon^{v+}_{kl}(\tau')\delta_{ij}\\ &-\int_{\tau'=0}^{t}\int_{\tau=0}^{t}\widetilde{\chi}_3(t,\tau,\tau')d\varepsilon^{v+}_{kk}(\tau)d\varepsilon^{v+}_{ij}(\tau') - \int_{\tau'=0}^{t}\int_{\tau=0}^{t}\widetilde{\chi}_4(t,\tau,\tau')d\varepsilon^{v+}_{ki}(\tau)d\varepsilon^{v+}_{kj}(\tau')\end{aligned} \tag{37}$$

2.3.3. Thermodynamic Conjugate of Damage Strengthening

The thermodynamic conjugate associated with the damage strengthening is determined through the differentiation of the instantaneous Helmholtz free energy with respect to β. Hence,

$$B = -\rho\frac{\partial\Psi}{\partial\beta} = -\hat{k}_D\beta \tag{38}$$

Equation (38) represents the associate conjugate to the strength to sustain damage, which is the state variable that controls the hardening or softening in the material due to the application of cyclic or sustained loading. The variables β and B control the resistance of the material to endure damage. Thus, $\boldsymbol{D}$, $\boldsymbol{Y}$, β, and B are coupled through the evolution equations.

2.3.4. Thermodynamic Conjugate of the Solidification Variable

Since Equation (35) is a function of the volume of solidified matter at the time where the load was applied, $v(\tau)$, it is straightforward to recognize that $V(t) = \rho\frac{\partial\Psi^{ve}}{\partial v(t)} = 0$. This indicates that the process of solidification has already stopped, such that the process vV is no longer active in the material at current time t. However, it is also interesting to determine the conjugate associated with $v(t)$ when the current time t approaches the loading time τ. This can be attained by computing the limit of $V(t)$ with respect to $v(t)$ when $t \to \tau$ (and $t \to \tau'$), i.e., $\lim V(t)_{t\to(\tau,\tau')} = \rho\frac{\partial\Psi}{\partial v(\tau)} + \rho\frac{\partial\Psi}{\partial v(\tau')}$, resulting in

$$\begin{aligned} V(\tau) = \rho\tfrac{\partial\Psi}{\partial v(\tau)} \quad & + \rho\tfrac{\partial\Psi}{\partial v(\tau')} \\ & = \tfrac{1}{2}\int_{\tau'=0}^{t}\int_{\tau=0}^{t}\lambda(2t-\tau-\tau')d\varepsilon_{ii}^{v}(\tau)d\varepsilon_{jj}^{v}(\tau') + \int_{\tau'=0}^{t}\int_{\tau=0}^{t}\mu(2t-\tau-\tau')d\varepsilon_{ij}^{v}(\tau)d\varepsilon_{ij}^{v}(\tau') \\ & + \int_{\tau'=0}^{t}\int_{\tau=0}^{t}\chi_1(2t-\tau-\tau')D_{ii}d\varepsilon_{jj}^{v+}(\tau)d\varepsilon_{kk}^{v+}(\tau') \\ & + \int_{\tau'=0}^{t}\int_{\tau=0}^{t}\chi_2(2t-\tau-\tau')D_{kk}d\varepsilon_{ij}^{v+}(\tau)d\varepsilon_{ij}^{v+}(\tau') \\ & + \int_{\tau'=0}^{t}\int_{\tau=0}^{t}\chi_3(2t-\tau-\tau')D_{ij}d\varepsilon_{kk}^{v+}(\tau)d\varepsilon_{ij}^{v+}(\tau') \\ & + \int_{\tau'=0}^{t}\int_{\tau=0}^{t}\chi_4(2t-\tau-\tau')D_{jk}d\varepsilon_{ij}^{v+}(\tau)d\varepsilon_{ik}^{v+}(\tau') + \tfrac{1}{2}\varepsilon_{ii}^{f}\varepsilon_{jj}^{f}\tfrac{\partial\hat{k}_f(\tau)}{\partial v(\tau)} + \tfrac{1}{2}\beta^2\tfrac{\partial\hat{k}_D(\tau)}{\partial v(\tau)} \end{aligned} \tag{39}$$

Equation (39) represents the energy stored due to the process of solidification at the moment in which the load is applied. However, according to Equation (39), it should be noted that, before loading occurs, the state of strain is considered to be equal to zero, and so is the damage strengthening, leading to $V(\tau) = 0$. This is because the process of solidification is chemical in nature rather than mechanical [45], and it does not depend on the state of strain as Equation (39) suggests. This generates inconsistency in the solidification theory; and therefore, a better assumption of the process of solidification must be studied in further research. However, it is noted that the definition of chemical energy due to the process of solidification is out of the scope of this paper.

2.4. Dissipative Processes

2.4.1. Dissipation Inequality

Let us substitute Equations (2) and (35) into the Clausius–Duhem inequality for uniform temperature and isothermal processes, i.e.,

$$\boldsymbol{\sigma}_{ij}\dot{\boldsymbol{\varepsilon}}_{ij}^{c} - \rho\dot{\Psi} \geq 0 \tag{40}$$

where $\dot{\varepsilon}_{ij}^{c}$ is the rate of creep strain and $\rho\dot{\Psi}$ is the rate of the thermodynamic potential, Equation (35). Hence, with the Leibniz rule, by considering the symmetry of the damage tensor, by incorporating Equations (16) and (23), and taking $\tau = \tau'$ in the terms with $\dot{\varepsilon}_{ij}^{v}$, and with Equations (36)–(40), this becomes

$$Y_{ij}\dot{D}_{ij} + s_{ij}\dot{\varepsilon}_{ij}^{f} + B\dot{\beta} + \Lambda \geq 0 \tag{41}$$

where $\dot{D}_{ij}$ are the components of the evolution of the damage tensor, s_{ij} are the components of the deviatoric stress, $\dot{\varepsilon}_{ij}^{f}$ is the rate of long-term creep strain, $\dot{\beta}$ is the evolution of damage strengthening, and Λ is the rate of dissipation energy given by

$$\begin{aligned} \Lambda = -\tfrac{1}{2}\int_{\tau'=0}^{t}\int_{\tau=0}^{t} \quad & \dot{\widetilde{\lambda}}(t,\tau,\tau')d\varepsilon_{ii}^{v}(\tau)d\varepsilon_{jj}^{v}(\tau') - \int_{\tau'=0}^{t}\int_{\tau=0}^{t}\dot{\widetilde{\mu}}(t,\tau,\tau')d\varepsilon_{ij}^{v}(\tau)d\varepsilon_{ij}^{v}(\tau') \\ & - \int_{\tau'=0}^{t}\int_{\tau=0}^{t}\dot{\widetilde{\chi}}_1(t,\tau,\tau')D_{ii}d\varepsilon_{jj}^{v+}(\tau)d\varepsilon_{kk}^{v+}(\tau') \\ & - \int_{\tau'=0}^{t}\int_{\tau=0}^{t}\dot{\widetilde{\chi}}_3(t,\tau,\tau')D_{ij}d\varepsilon_{kk}^{v+}(\tau)d\varepsilon_{ij}^{v+}(\tau') \\ & - \int_{\tau'=0}^{t}\int_{\tau=0}^{t}\dot{\widetilde{\chi}}_4(t,\tau,\tau')D_{jk}d\varepsilon_{ij}^{v+}(\tau)d\varepsilon_{ik}^{v+}(\tau') - \tfrac{1}{2}\dot{\hat{k}}_f\varepsilon_{ii}^{f}\varepsilon_{jj}^{f} - \tfrac{1}{2}\dot{\hat{k}}_D\beta^2 \end{aligned} \tag{42}$$

where $\dot{\widetilde{\lambda}}$ and $\dot{\widetilde{\mu}}$ are the rate of the first and second Lamé aging relaxation functions $\widetilde{\lambda}$ and $\widetilde{\mu}$, Equation (16), respectively; $\dot{\widetilde{\chi}}_I$ (with $I = 1, 2, \cdots, 4$) are the rate of the aging relaxation functions $\widetilde{\chi}_I$, Equation (23); $\dot{\hat{k}}_f$ is the rate of the bulk aging function $\hat{k}_f$; and $\dot{\hat{k}}_D$ is the rate of the aging hardening function $\hat{k}_D$.

2.4.2. Restrictions Imposed by the First and Second Laws of Thermodynamics

It is necessary for the energy stored in the material to be semi-positive definite, in the case of viscoelastic materials [54], i.e., $\rho\Psi \geq 0$. In addition, the rate of energy dissipation must also be semi-positive definite, i.e., $\Lambda \geq 0$, according to Equation (41). From these two

restrictions, by considering the damage variable as a semi-positive definite tensor, and by stating that each term of Equations (35) and (42) must meet these inequalities, one finds

$$\begin{gathered}\widetilde{\lambda}(t,\tau,\tau') \geq 0 \\ \widetilde{\mu}(t,\tau,\tau') \geq 0 \\ \widetilde{\chi}_I(t,\tau,\tau') \geq 0 \text{ (with } I = 1,2,\cdots,4) \\ \hat{k}_f \geq 0 \\ \hat{k}_D \geq 0\end{gathered} \tag{43}$$

and

$$\begin{gathered}\dot{\widetilde{\lambda}}(t,\tau,\tau') \leq 0 \\ \dot{\widetilde{\mu}}(t,\tau,\tau') \leq 0 \\ \dot{\widetilde{\chi}}_I(t,\tau,\tau') \leq 0 \text{ (with } I = 1,2,\cdots,4) \\ \dot{\hat{k}}_f \leq 0 \\ \dot{\hat{k}}_D \leq 0\end{gathered} \tag{44}$$

Hence, Equations (43) and (44) are the thermodynamic restrictions imposed by the first and second laws of thermodynamics to the aging relaxation functions and the functions $\hat{k}_f$ and $\hat{k}_D$. These restrictions must be met in the time domain $t \geq 0$. Furthermore, in the case of $\hat{k}_f$ and $\hat{k}_D$, the restrictions show that they do not only depend on the solidification process $v(t)$; and in fact, the material keeps hardening as time advances [43,45]. This is because $v(t)$ is a positive function with a positive change rate. The same implication applies to the relaxation functions $\widetilde{\lambda}$, $\widetilde{\mu}$, and $\widetilde{\chi}_I$, which was already stated in Equations (12), (13), (16), (23), and (24).

2.4.3. Evolution Equations

To determine the evolution equations, the relationship between thermodynamic fluxes and forces using a dissipation potential [58,59] is considered, i.e., $\boldsymbol{J} = \dot{\xi}\frac{\partial \Omega}{\partial \boldsymbol{X}}$, where $\boldsymbol{J}$ are the thermodynamic fluxes, $\boldsymbol{X}$ are the thermodynamic forces, Ω is the dissipation potential, and $\dot{\xi}$ is a multiplier that needs to be determined from the consistency condition. Hence, from the first three terms in Equation (41), the thermodynamic forces are Y_{ij}, B, and s_{ij}, while the thermodynamic fluxes are $\dot{D}_{ij}$, $\dot{\beta}$, and $\dot{\varepsilon}^f_{ij}$. If two dissipation inequalities are considered, one related to the damage process and the other related to the viscous flow, i.e.,

$$\Phi^D + \Phi^f \geq 0 \tag{45}$$

where $\Phi^D = Y_{ij}\dot{D}_{ij} + B\dot{\beta}$, and $\Phi^f = s_{ij}\dot{\varepsilon}^f_{ij}$, then each term in Equation (45) must satisfy the inequality, i.e., $\Phi^D \geq 0$ and $\Phi^f \geq 0$. This is valid since the viscous flow occurs at any level of stress (or strain), without damage. In addition, the strength to sustain damage (i.e., hardening) must be coupled with the damage experienced by the material, as previously stated.

To account for the long-term creep strain of the uniaxial model and the viscous flow, $\sigma = \hat{\eta}(t)\dot{\varepsilon}^f$ is implemented in [43], where $\hat{\eta}(t)$ is the aging viscosity function given by $\hat{\eta}(t) = \eta_0 v(t)$, with η_0 known as the effective viscosity of the solidified matter. Following the same line of thought [57],

$$s_{ij} = \hat{\eta}(t)\dot{\varepsilon}^f_{ij} \tag{46}$$

Equation (46) provides the evolution equation that relates the thermodynamic force s_{ij} and the thermodynamic flux $\dot{\varepsilon}^f_{ij}$, through the multiplier $\hat{\eta}$. Furthermore, if Equation (46) is substituted into $\Phi^f = s_{ij}\dot{\varepsilon}^f_{ij}$, the inequality $\Phi^f \geq 0$ is satisfied. Furthermore, the restriction

of the second law of thermodynamics is imposed to the viscosity as $\hat{\eta}(t) \geq 0$, which implies that $v(t) \geq 0$, and by differentiating both sides with respect to t, one finds $\dot{v}(t) \geq 0$. This is consistent with the positive nature of both $v(t)$ and its rate of change, as described earlier [43].

For the damage and damage strengthening evolution,

$$\begin{aligned} \dot{D}_{ij} &= \dot{\xi}^{D} \frac{\partial \Omega^{D}}{\partial Y_{ij}} \\ \dot{\beta} &= \dot{\xi}^{D} \frac{\partial \Omega^{D}}{\partial B} \end{aligned} \tag{47}$$

where Ω^D is the dissipation potential and $\dot{\xi}^{D}$ is a multiplier defined through the consistency condition. Ω^D and $\dot{\xi}^{D}$ must be considered in both equations, Equation (47), to couple damage and the strength to sustain damage. In general, the dissipation potential Ω^D has been defined as a loading surface f^D, which in turn is defined as a function of Y_{ij} and B [25,29]. Hence, the loading surface,

$$\Omega^{D} = f^{D} = Y_{EQ} - (B + B_o) \leq 0 \tag{48}$$

where Y_{EQ} is the equivalent energy release rate, defined as $Y_{EQ} = \left[\frac{1}{2} Y_{ij} Y_{ij}\right]^{\frac{1}{2}}$, B is the damage strengthening, and B_o is a constant that defines the threshold value for cracks to start growing inside the material. This former assumption is valid since the onset of cracks occurs when a certain value of the stress is attained. For instance, in specimens subjected to compression, the onset occurs when the stress reaches about 45 to 50 percent of the compressive strength, i.e., $0.45f'c - 0.50f'c$. It is important to note that Equation (48) was defined through an analogy to plasticity theory [25,29,60]. Hence, through the substitution of Equation (48) into Equation (47), one obtains

$$\begin{aligned} \dot{D}_{ij} &= \frac{\dot{\xi}^{D}}{2Y_{EQ}} Y_{ij} \\ \dot{\beta} &= -\dot{\xi}^{D} \end{aligned} \tag{49}$$

Furthermore, as mentioned above, the multiplier $\dot{\xi}^{D}$ is defined through the consistency condition $\dot{f}^{D}\left(Y_{ij}, B\right) = 0$. Then, with the chain rule,

$$\dot{f}^{D} = \frac{1}{2Y_{EQ}} Y_{ij} \dot{Y}_{ij} - \dot{B} = 0 \tag{50}$$

This is provided that the rate of change in Equation (38) is obtained and that the second equation in Equation (49) is integrated with respect to time t. Then, if both these conditions are incorporated into Equation (50), and through some manipulations, one finds

$$\hat{k}_D \dot{\xi}^{D} + \dot{\hat{k}}_D \xi^{D} = \frac{1}{2Y_{EQ}} Y_{ij} \dot{Y}_{ij} + C_o \hat{k}_D \tag{51}$$

where C_o is a constant of integration, which is obtained from the integration of the second term of Equation (49); and it is defined from experimental measurements of the strength to sustain damage (hardening or softening) at a given time t_o. It is easy to recognize that the left-hand side of Equation (51) is the product rule of differentiation, i.e., $u\dot{v} + v\dot{u} = f$. Hence, through some manipulations, ξ^D and its rate $\dot{\xi}^{D}$ are obtained, respectively,

$$\begin{aligned} \xi^{D} &= \frac{1}{\hat{k}_D} \int \frac{1}{2Y_{EQ}} Y_{ij} \dot{Y}_{ij} dt + C_1 \frac{1}{\hat{k}_D} + C_o \\ \dot{\xi}^{D} &= \frac{1}{2\hat{k}_D Y_{EQ}} Y_{ij} \dot{Y}_{ij} - \frac{\dot{\hat{k}}_D}{\hat{k}_D^2} \left\{ \int \frac{1}{2Y_{EQ}} Y_{ij} \dot{Y}_{ij} dt + C_1 \right\} \end{aligned} \tag{52}$$

where C_1 is also a constant of integration, which is defined from experimental measurements of the strength to sustain damage (hardening or softening) at a given time t_1. It should be noted that, if $\hat{k}_D$ is a positive constant, i.e., $\hat{k}_D = k_{do}$, the multiplier $\dot{\xi}^D$, second equation in Equation (52), is simplified as $\dot{\xi}^D = \frac{1}{2k_{do}Y_{EQ}} Y_{ij}\dot{Y}_{ij}$. This situation is plausible since, as stated above, the onset of cracks is considered for a specific value of stress, without assuming hardening due to the solidification process. Therefore, it can be further assumed that, even if the material hardens with time, the hardening attained due to the solidification process $v(t)$ does not vary substantially with respect to those values obtained at 28 days specifically for the strengthening process βB, and thus, the process of solidification associated to $\hat{k}_D$ becomes stationary. By assuming $\hat{k}_D$ as a positive constant k_{do}, the thermodynamic restrictions in Equations (43) and (44) are also satisfied. Nevertheless, the second term of Equation (52) provides the complete solution from which the constitutive equations and their respective evolution equations are satisfied. Let us finally assume a loading–unloading condition, in accordance with [29], with the aim of obtaining damage due to cyclic loading; hence,

$$\begin{aligned} \dot{\xi}^D &= 0 \text{ when } \tfrac{\partial f^D}{\partial Y_{kl}}\dot{Y}_{kl} \leq 0 \text{ (unloading)} \\ \dot{\xi}^D &= \tfrac{1}{2\hat{k}_D Y_{EQ}} Y_{ij}\dot{Y}_{ij} - \tfrac{\dot{\hat{k}}_D}{\hat{k}_D^2}\left\{\int \tfrac{1}{2Y_{EQ}} Y_{ij}\dot{Y}_{ij}dt + C_1\right\} \text{ when } \tfrac{\partial f^D}{\partial Y_{kl}}\dot{Y}_{kl} > 0 \text{ (loading)} \end{aligned} \tag{53}$$

Equation (53) provides the means of analyzing reversible loading. The first equation represents the unloading, which means that the stress is decreasing or has reached its peak value. On the other hand, the second equation represents the loading, i.e., increasing stress value.

Both equations in Equation (49), with Equation (53), are the evolution equation that relates the thermodynamic forces Y_{ij} and B, to the thermodynamic fluxes $\dot{D}_{ij}$ and $\dot{\beta}$, respectively.

The constitutive equations, from Equations (36) to (38), along with the evolution equations, Equations (46) and (49) with Equation (53), the relaxation functions in Equations (16) and (23), and thermodynamic restrictions in Equations (43) and (44), provide the complete set of equations of the non-linear creep relaxation model with anisotropic damage developed.

2.5. Form of the Non-Aging Relaxation Functions

Finally, the form of the non-aging relaxation functions is discussed. The mathematical form of the basic (non-aging) relaxation function can be obtained using functional equations. The form of the relaxation functions is determined through the second Cauchy equation. It is worth mentioning that other forms of equations can be used (such as the first Cauchy equation). However, those equations do not satisfy the thermodynamic restrictions expressed in Equations (43) and (44). If one recalls that $v(t) \geq 0$ and $\dot{v}(t) \geq 0$, the relaxation functions in Section 2.3.1 must also meet the thermodynamic restrictions in Equations (43) and (44), i.e.,

$$\begin{aligned} \lambda(t-\tau) &\geq 0 \\ \mu(t-\tau) &\geq 0 \\ \chi_I(t-\tau) &\geq 0 \text{ (with } I = 1, 2, \cdots, 4) \end{aligned} \tag{54}$$

and

$$\begin{aligned} \dot{\lambda}(t-\tau) &\leq 0 \\ \dot{\mu}(t-\tau) &\leq 0 \\ \dot{\chi}_I(t-\tau) &\leq 0 \text{ (with } I = 1, 2, \cdots, 4) \end{aligned} \tag{55}$$

From the first statement of the proof of the second Cauchy equation, i.e., $f(t+\tau) = f(t)f(\tau)$, for all t, τ that belongs to the real numbers, and where $f(t)$ is a function of real

numbers that maps to the real numbers, the similarity with Equations (54) and (55) is recognized. Hence, it is easy to prove [61] that $f(t) = a^t$ or $f(t) = e^t$, and thus, through the thermodynamic restrictions,

$$\begin{aligned} \lambda(t) &= \sum_{i=1}^{n} C_{\lambda i} e^{-k_i^{\lambda} t} \\ \mu(t) &= \sum_{i=1}^{n} C_{\mu i} e^{-k_i^{\mu} t} \\ \chi_I(t) &= \sum_{i=1}^{n} C_{\chi_{Ii}} e^{-k_i^{\chi I} t} \text{ (with } I = 1, 2, \cdots, 4) \end{aligned} \tag{56}$$

The summation was considered for a general solution to represent the Maxwell chain [41,44,59], where e is Euler's number; k_i^{λ}, k_i^{μ}, and $k_i^{\chi I}$ are a set of positive experimental constants associated with the non-aging relaxation function λ, μ, and χ_I with $I = 1, 2, \cdots, 4$, respectively; while $C_{\lambda i}$, $C_{\mu i}$, and $C_{\chi_{Ii}}$ are experimental coefficients associated to their corresponding relaxation function, added to shift the relaxation functions. Equation (56) is consistent with the Dirichlet series used in [41,44]. The graphical representation of Equation (56) is consistent with recent experimental data obtained [62]. It should be mentioned that the variable t is replaced by the loading duration $(t - \tau)$ when the relaxation functions are substituted into the constitutive model, as shown in Equations (54) and (55).

3. Application of the Constitutive Model and Discussion of Results

To determine the applicability of the constitutive model developed, two theoretical examples are presented in this section. The first is a constant uniaxial state of strain to prove the relaxation capabilities of the model. The second is associated with tertiary creep, to prove the temporal nature of the damage in the model. It is important to mention that both examples only quantify the short-term creep strain. This is mainly because the long-term creep strain is consistent with [43], where its applicability has already been established. Furthermore, $\hat{k}_D = k_{do}$ is assumed in both cases to simplify the solution of the equations.

3.1. Relaxation Case Study

A particle subjected to a constant uniaxial strain, as shown in Figure 6, is given by $\varepsilon_{11}^{v} = \varepsilon_0 H_t$ (and $\varepsilon_{ij}^{v} = 0$, otherwise) through the Heaviside function H_t, and the immediate strain can be considered. It is easily proven that the positive orthogonal projection components are $P_{1111}^{\varepsilon+} = 1$ and $P_{ijkl}^{\varepsilon+} = 0$, otherwise. Thus, the positive strain tensor is equal to the strain tensor, i.e., $\varepsilon_{ij}^{v+} = \varepsilon_{ij}^{v}$. Then, by substitution into the constitutive model equations, Equations (36)–(38), (46), (49), and (53), and the relaxation functions in Equation (56), the field equations that describe the relaxation case study are determined by assuming the following constant values for simplification: $v(0) = 1$, $\varepsilon_0^{v} = 1$, $k_{do} = 20$, $C_{\lambda i} = C_{\mu i} = C_{\chi_{Ii}} = 1$, $k_i^{\lambda} = k_i^{\mu} = k_i^{\chi I} = k_0 = \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1$. The graphical behavior of the state variables is shown in Figures 7 and 8. It is worth mentioning that the first equation in Equation (53) was not considered to show the time-dependent (rate-type) capabilities of the model.

Figure 7a,b shows the relaxation of the state of stress, and the behavior is typical of a viscoelastic material. The parameter σ_{11} represents the stress due to the application of the short-term creep strain ε_{11}^{v}, while the stresses σ_{22} and σ_{33} are developed due to the confinement of the specimen, as shown in Figure 6. On the other hand, Figure 7c,d shows the active damage variables in the material. As shown in Figure 7c, the maximum damage occurs in the direction where the strain is applied, which is consistent with the type of boundary conditions specified. Figure 7d shows that damage also occurs in the perpendicular directions of the strain. This is because the constitutive equations consider a coupling in damage in other directions. Although the micro-cracks open and grow driven by the tensile strains, in general, the micro-cracks are not perfectly planar in nature. This feature was considered first by Ortiz [8]. Nonetheless, the developed model considers this feature as a natural consequence of the representation theorem utilized to define the

thermodynamic potential, Equation (35). It is worth noting that a healing mechanism develops inside the material as time advances, as shown in Figure 7c,d. This behavior is due to the fact that the unloading process is being analyzed. Although this should be omitted in the calculations, the obtained results are relevant, since they agree with the definition of damage in viscoelastic materials, in which the damage is time-dependent [43]. Furthermore, from Figure 8e, the loading–unloading condition takes a positive value only when $t = 0$, and negative or zero when $t > 0$, and thus, damage only grows when the loading–unloading condition is positive definite. Figure 8a–d are consistent with the damage and stress behavior shown.

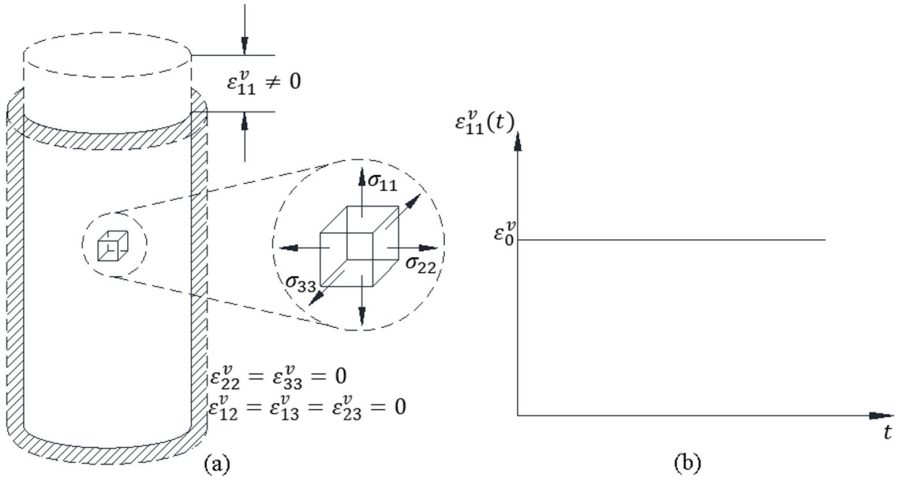

Figure 6. State of strain considered; (**a**) Specimen of study; (**b**) Constant strain at any given time $t > 0$.

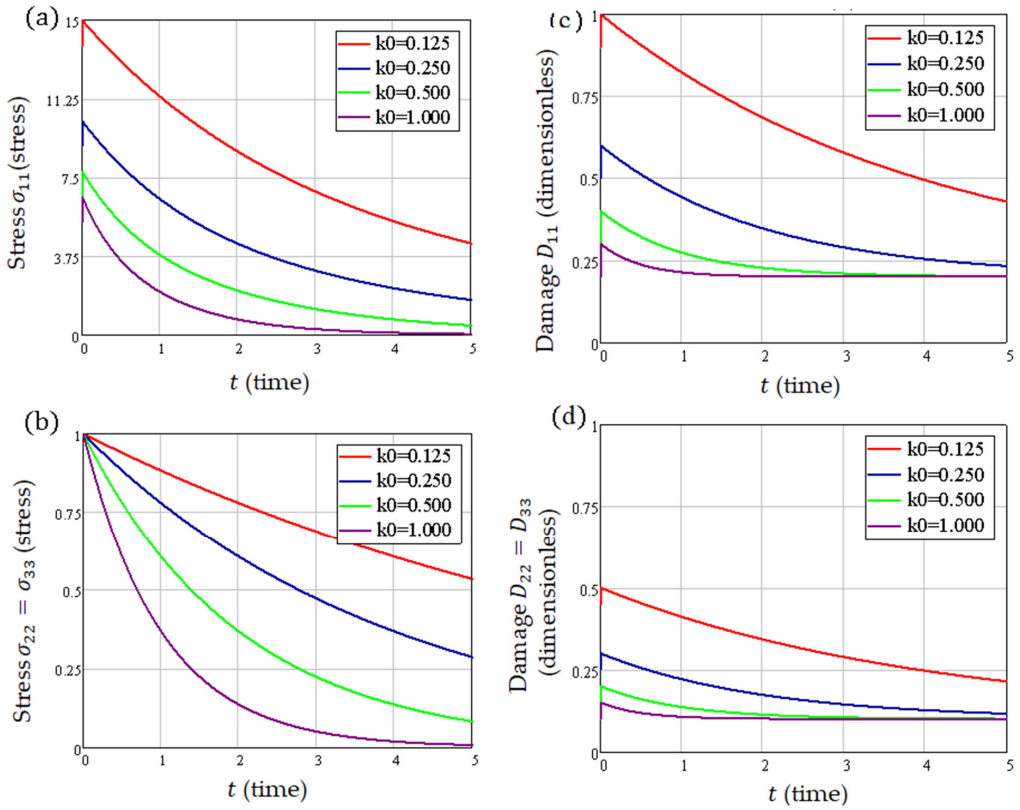

Figure 7. State of stress and damage variables for the relaxation case study; (**a**) Stress in the direction of the acting strain σ_{11}; (**b**) Stresses acting perpendicular to the acting strain σ_{22} and σ_{33}; (**c**) Damage variable in the direction of the acting strain D_{11}; (**d**) Damage variable acting perpendicular to the acting strain D_{22} and D_{33}.

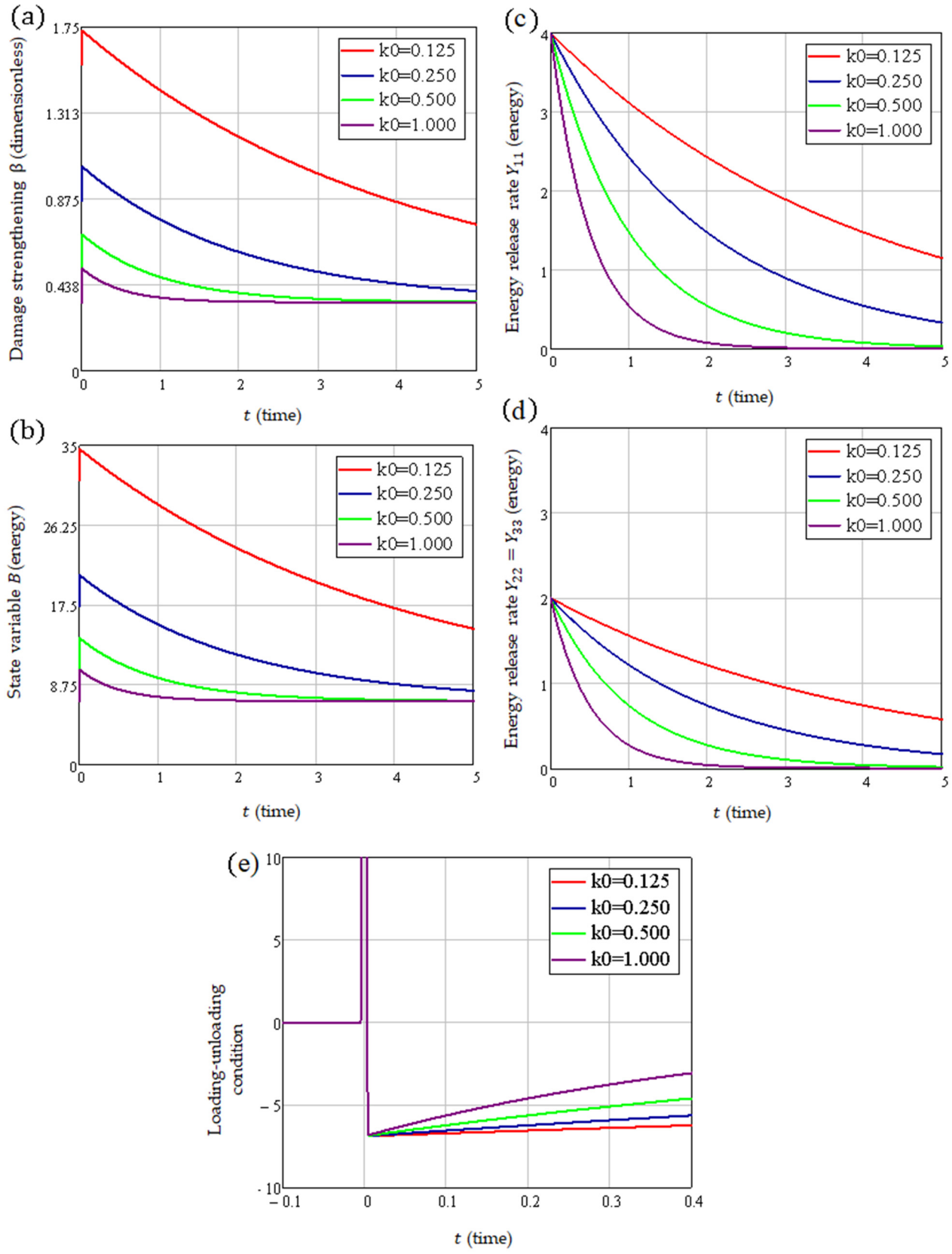

Figure 8. Damage strengthening and its associated variable, Energy release rate, and loading condition for the relaxation case study; (**a**) Damage strengthening; (**b**) Variable B; (**c**) Energy release rate Y_{11}; (**d**) Energy release rate Y_{22} and Y_{33}; (**e**) Loading condition.

3.2. Creep Case Study

In a similar manner, a particle subjected to a constant uniaxial stress is considered, as shown in Figure 9. However, in this case, to avoid solving the non-linear Volterra integral equations that define the constitutive model, a growing function of strain is considered, aiming to obtain a similar behavior as the one shown in Figure 9. Hence, if $\sigma_{11} > 0$, it is expected that the strains are $\varepsilon_{11}^{v} > 0$, $\varepsilon_{22}^{v} < 0$, $\varepsilon_{33}^{v} = \varepsilon_{22}^{v}$, and $\varepsilon_{ij}^{v} = 0$, otherwise. This consideration results in $P_{1111}^{\varepsilon+} = 1$ and $P_{ijkl}^{\varepsilon+} = 0$, otherwise. Thus, $\varepsilon_{11}^{v+} = \varepsilon_{11}^{v}$, and $\varepsilon_{ij}^{v+} = 0$, otherwise.

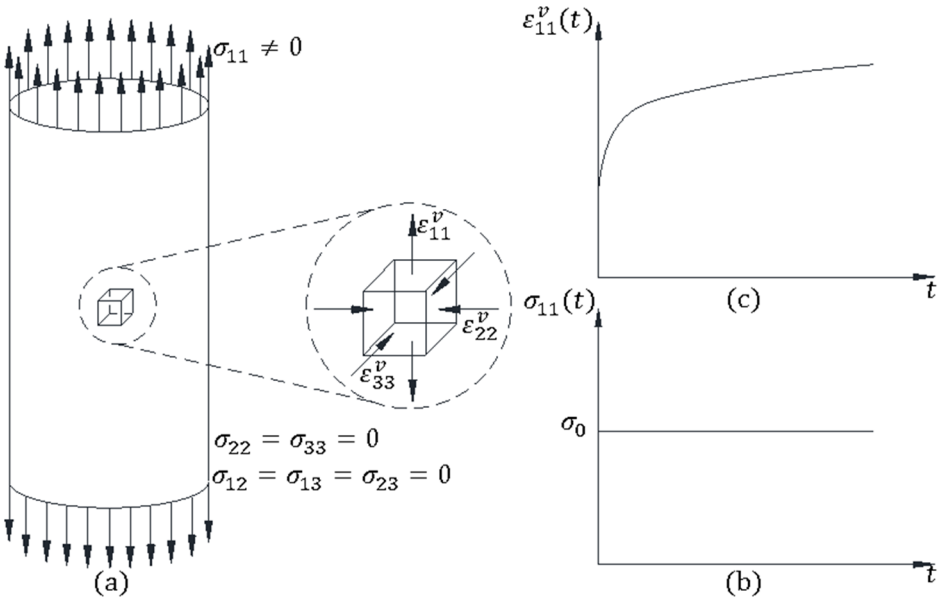

Figure 9. State of stress considered; (**a**) Specimen of study; (**b**) Constant stress at any given time $t > 0$; (**c**) Short-term creep strain experienced in the material.

If stresses and strains are substituted into the constitutive equations of the model, by some simplifications through the Laplace transform, the stress–strain relation in its Riemann–Stieltjes form is obtained as follows:

$$\sigma_{11} = \int_{\tau=0}^{t} \hat{E}(t,\tau) \quad \frac{\partial \varepsilon_{11}^{v}}{\partial \tau} d\tau + 2\int_{\tau=0}^{t} \hat{\chi}_1(t,\tau) D_{kk} \frac{\partial \varepsilon_{11}^{v}}{\partial \tau} d\tau + 2\int_{\tau=0}^{t} \hat{\chi}_2(t,\tau) D_{mm} \frac{\partial \varepsilon_{11}^{v}}{\partial \tau} d\tau + 2\int_{\tau=0}^{t} \hat{\chi}_3(t,\tau) D_{11} \frac{\partial \varepsilon_{11}^{v}}{\partial \tau} d\tau + 2\int_{\tau=0}^{t} \hat{\chi}_4(t,\tau) D_{11} \frac{\partial \varepsilon_{11}^{v}}{\partial \tau} d\tau \tag{57}$$

where $\hat{E}(t,\tau)$ is the uniaxial aging relaxation function defined in [4,5] as $\hat{E}(t,\tau) = v(\tau)E(t-\tau)$. This uniaxial aging relaxation function was obtained by considering the relation between the aging Lamé relaxation functions $\hat{\lambda}$ and $\hat{\mu}$, with $\hat{E}$ through the Laplace transform, in a similar manner as in [54]. Furthermore, it is important to state that, if the damage variable is equal to zero, Equation (57) reduces to the linear aging viscoelastic stress–strain relation under uniaxial stress, given in [43]. In addition, the relationship between the strain ε_{11}^{v} and ε_{22}^{v} is given through the Laplace transform,

$$\bar{\hat{\varepsilon}}_{22}^{v} = -s\bar{v}\bar{\hat{\varepsilon}}_{11}^{v} \tag{58}$$

where $\bar{\hat{\varepsilon}}_{22}^{v}$ is the Laplace transform of the aging strain $\hat{\varepsilon}_{22}^{v}$ defined as $\hat{\varepsilon}_{22}^{v} = v(\tau)\frac{\partial \varepsilon_{22}^{v}(\tau)}{\partial \tau}$.For manipulation of the Laplace transform of the convolution integral equation, $\bar{\hat{\varepsilon}}_{11}^{v}$ is the Laplace transform of the aging strain $\hat{\varepsilon}_{11}^{v}$, defined similarly as $\hat{\varepsilon}_{11}^{v} = v(\tau)\frac{\partial \varepsilon_{11}^{v}(\tau)}{\partial \tau}$; $\bar{v}$ is the Laplace transform of Poisson relaxation function; and s is the Laplace parameter. It is worth mentioning that the aging Poisson relaxation function $\hat{v}$ was obtained through the relation

between Lamé relaxation functions $\hat{\lambda}$ and $\hat{\mu}$, with $\hat{\nu}$, similarly as for $\hat{E}$. The remaining state variables are obtained by substitution as in the relaxation case study.

For simplification, we assume the strain $\varepsilon_{11}^{v} = \varepsilon_0 H_t(t)\,[1 - \frac{1}{3}e^{-3k_0 t}]$, and also assume $v(\tau) = v(\tau') = 1$, $\frac{\varepsilon_0}{\sigma_0} = \frac{1}{3}$, $k_{do} = 1$, $C_{\lambda i} = C_{\mu i} = C_{\chi_{Ii}} = 1$, $k_i^{\lambda} = k_i^{\mu} = k_i^{\chi I} = k_0 = \frac{1}{10}$. The graphical behavior of some of the state variables is shown in Figure 10. It is worth mentioning that the first equation in Equation (53) was not considered to show the time-dependent (rate-type) capabilities of the model.

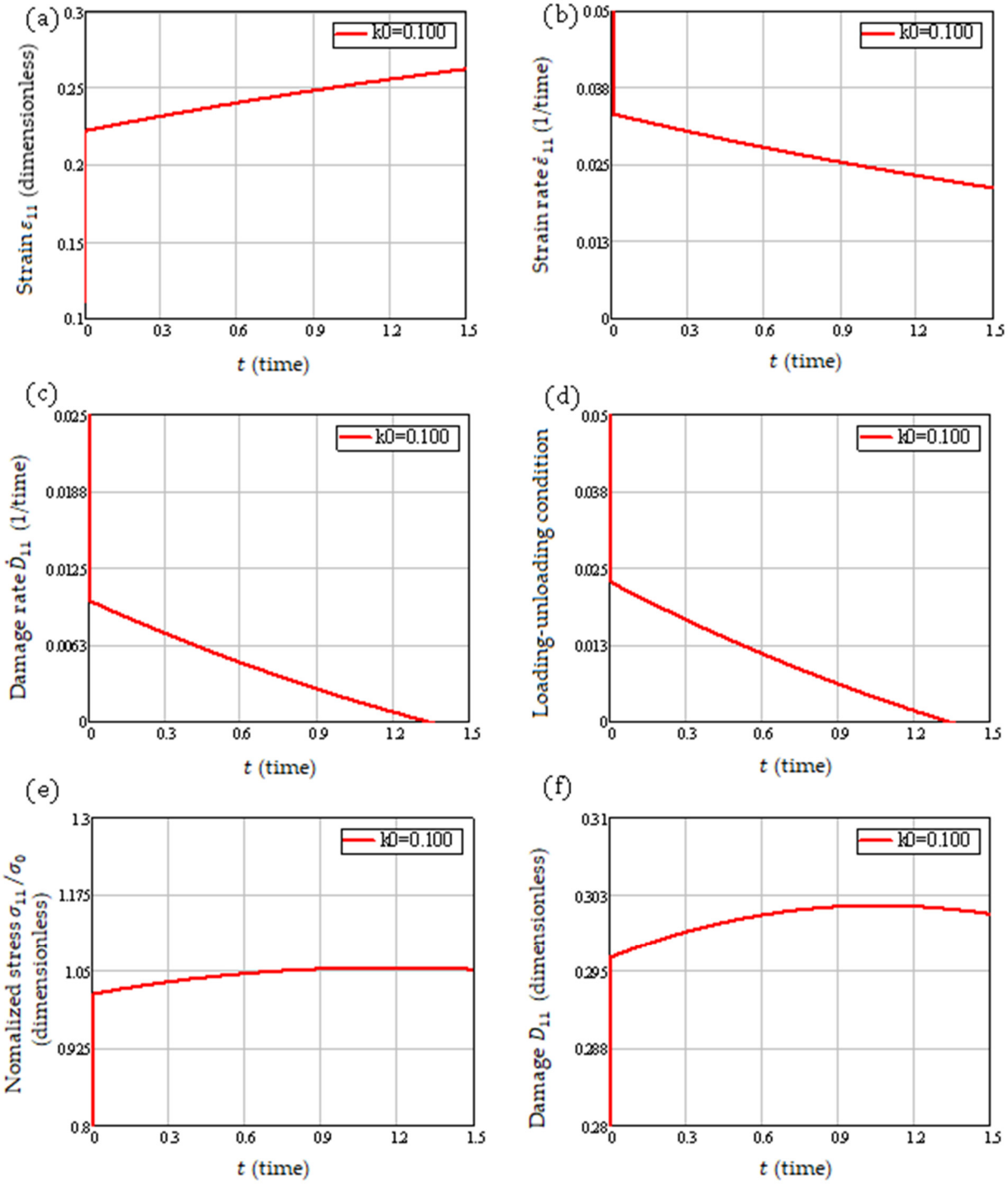

Figure 10. Graphical representation of some of the state variables for the creep case study; (**a**) Strain ε_{11}^{v}, (**b**) Rate of strain $\dot{\varepsilon}_{11}^{v}$; (**c**) Damage rate $\dot{D}_{11}$; (**d**) Loading–unloading condition $\frac{\partial f^D}{\partial Y_{ij}}\dot{Y}_{ij}$; (**e**) Normalized state of stress $\frac{\sigma_{11}}{\sigma_0}$; (**f**) Damage variable D_{11}.

From Figure 10d,f, it can be concluded that the damage continues to grow as long as the loading–unloading condition remains positive. This effect is consistent with what is expected in tertiary creep, in which the micro-cracks continue growing as time advances. Figure 10f proves that the model developed is suitable for analyzing tertiary creep, as discussed earlier. On the other hand, in regard to the stress behavior, from Figure 10e, it can be concluded that the strain response is not the exact solution of the non-linear Volterra integral equation, Equation (57), under constant sustained stress, i.e., $\sigma_{11} = \sigma_0 H_t(t)$. As observed in Figure 10e, the response of the stress obtained is close to the straight line expected, which implies that the strain is not the exact solution to Equation (57). Nevertheless, the behavior is not changing substantially as time advances, leading to a fair approximation. From this behavior, we can infer that, if the creep response is provided, the material will respond as expected for a viscoelastic damageable material. Moreover, it is important to recall that the analysis performed shall be stopped at the moment that the loading–unloading condition reaches the value of zero since no further damage develops. If this analysis is not stopped when that occurs, the material will be enduring the healing mechanism already described.

4. Conclusions

A thermodynamic constitutive model has been formulated based on the Bažant theory of solidification and the theory of continuum damage mechanics. The model is strain-based and can predict the anisotropy of the material due to the microcracks growth and the time-dependent damage (known as tertiary creep). To develop the formulation, several steps and considerations were adopted. These steps and considerations, along with their advantages and shortcomings, are listed next. These also serve to point out the main conclusion of this research.

1. A second-order tensor damage variable was considered to account for the anisotropy induced. The principal directions of the damage variable are assumed to coincide with those of the viscoelastic strain tensor. No initial damage was considered. Therefore, the formulation herein is not suitable for damage presented before strain occurs, which is the case of cracking due to drying shrinkage, poor curing, thermal, and others.
2. The solidification is assumed to be independent of the position vector $\boldsymbol{x}$. This is consistent with the theory of solidification of basic creep proposed by Bažant. The solidification process is dependent on the position. The conjugate of the solidification process was found to be zero when no strain had been applied. This is because the solidification process is chemical in nature.
3. The creep strain was separated into two main components, viscoelastic strain and viscous flow strain, consistently with the Bažant rheological model. The immediate strain is not considered. However, depending on the definition of the input strain, the immediate strain can be considered.
4. The thermodynamics functions of aging viscoelasticity were extended to a three-dimensional case coupled with anisotropic damage, by assuming the material as viscoelastic damageable material (which is analogous to the theory for brittle elastic materials in continuum damage mechanics) and the aid of a representation theorem.
5. The relationship between the strength to sustain damage and its conjugate variable was considered linear, which is consistent with the assumption of the damage. However, experimental results are needed to define whether the proportionality function $\hat{k}_{\boldsymbol{D}}$ can be assumed as a constant k_{do}.
6. The constitutive equations were determined from the thermodynamic potential formulas. The dissipation inequality was found from the Clausius–Duhem inequality along with the thermodynamic potential. The dissipation inequality found shows three dissipating processes: damage state, damage strengthening, and viscous flow. The damage and damage strengthening were considered as coupled, while the viscous flow is assumed as independent since it occurs at any level of strain.

7. The determination of the evolution equations was based on the loading surface as a dissipation potential assumption. The potential for the damage dissipative process was assumed to be similar to the "potential theory" in the theory of plasticity, which has been proven suitable for concrete. A loading–unloading condition was considered, which leads to the possibility of analyzing tertiary creep.
8. The thermodynamic restrictions on the relaxation function were obtained based on the positive energy assumption and the Clausius–Duhem inequality. The former governs the semi-positive definite form of the relaxation functions, while the latter leads to the negative (decreasing behavior) rate of change of the relaxation functions.
9. The second Cauchy equation of functional equations is suitable for the thermodynamic restrictions obtained. In addition, the second Cauchy equation is equivalent to the Dirichlet series utilized in the development of the relaxation functions.
10. To determine the applicability of the formulation, two practical case studies were carried out. From the graphical representation, we can conclude that the formulation developed can account for time-dependent (rate-type) damage as long as the loading–unloading condition remains positive.

Author Contributions: Conceptualization, B.T.T.-T., S.E.N. and J.M.; methodology, B.T.T.-T., J.M. and S.E.N.; validation, B.T.T.-T., S.E.N. and J.M.; formal analysis, I.F.-V., J.M.M.-R. and C.A.J.-A.; writing—original draft preparation, B.T.T.-T. and C.A.J.-A.; writing—review and editing, I.F.-V. and J.M.M.-R. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The data presented in this study are openly available at https://repository.iit.edu/islandora/object/islandora:8838.

Acknowledgments: The authors would like to thank the Facultad de Ingeniería Civil (FIC) and the Instituto de Ingeniería Civil (IIC) of the Universidad Autónoma de Nuevo León (UANL), especially Luis Manuel Aranda and Pedro L. Valdez Tamez, for their support. Also, the authors would like to thank the Civil, Architectural, and Environmental Engineering (CAEE) department at the Illinois Institute of Technology (IIT), for all their support. The help provided by the late Domingo Carreira of Illinois Institute of Technology, during the course of the research for the dissertation of the lead author, is gratefully acknowledged.

Conflicts of Interest: The authors declare no conflicts of interest.

Nomenclature

B	Thermodynamic conjugate of the damage strengthening;
$\dot{B}$	Rate of the thermodynamic conjugate of the damage strengthening;
B_o	Threshold value for cracks to start growing inside the material;
C_o	Constant of integration defined from experiments on strength damage;
C_1	Constant of integration defined from experiments on strength damage;
$C_{\lambda i}$	Experimental coefficients associated with λ non-aging function;
$C_{\mu i}$	Experimental coefficients associated with μ non-aging function;
$C_{\chi_{1i}}$	Experimental coefficients associated with χ_I non-aging function;
D_i	Eigenvalues of the damage tensor;
D_{ij}	Damage tensor components;
$\dot{D}_{ij}$	Rate of the damage tensor components;
$\boldsymbol{D}$	Damage tensor;
e_{ij}^{v}	Components of deviatoric part of short-term creep strain;
f^D	Loading surface for damage;
$\dot{f}^D$	Consistency condition for damage;
G_1	Isotropic age-independent relaxation function related to the deviatoric strain;
G_2	Isotropic age-independent relaxation function related to the dilatational strain;

$\hat{G}_1$	Isotropic age-dependent relaxation function related to the deviatoric strain;
$\hat{G}_2$	Isotropic age-dependent relaxation function related to the dilatational strain;
G_{ijkl}	Components of age-independent relaxation tensor;
H_t	Temporal Heaviside function;
H_ε	Heaviside function defined in the principal strain space;
$\boldsymbol{I}$	Identity tensor;
$\boldsymbol{J}$	Thermodynamic fluxes;
k_0	Experimental exponent to replace k_i^λ, k_i^μ and $k_i^{\chi I}$ for simplification;
k_{do}	Constant replacing the function $\hat{k}_D$ for simplification;
k_i^λ	Experimental exponents of the λ non-aging function;
k_i^μ	Experimental exponents of the μ non-aging function;
$k_i^{\chi I}$	Experimental exponents of the χ_I non-aging function;
$\hat{k}_D$	Experimental aging hardening function;
$\dot{\hat{k}}_D$	Rate of change in experimental aging hardening function;
$\hat{k}_f$	Long-term creep bulk aging function;
$\dot{\hat{k}}_f$	Rate of change in long-term creep bulk aging function;
$\boldsymbol{n}^{(i)D}$	Eigenvectors of the damage tensor;
$\boldsymbol{n}^{(i)\varepsilon}$	Eigenvectors of the strain tensor;
p	Hydrostatic pressure;
$\mathbb{P}^{\varepsilon+}$	Positive orthogonal projection of the strain;
$\boldsymbol{Q}^{\varepsilon+}$	Positive strain rotation tensor;
s_{ij}	Deviatoric stress tensor components;
$\boldsymbol{s}$	Deviatoric stress tensor;
t	Current time variable;
v	Degree of solidification;
V	Thermodynamic conjugate of the degree of solidification;
$\boldsymbol{x}$	Position vector;
$\boldsymbol{X}$	Thermodynamic forces;
Y_{EQ}	Equivalent energy release rate;
Y_{ij}	Energy release rate components;
$\boldsymbol{Y}$	Energy release rate associated to the damage;
β	Damage strengthening;
$\dot{\beta}$	Rate of the damage strengthening;
δ_{ij}	Kronecker delta;
ε_{ij}	Strain tensor components;
$\boldsymbol{\varepsilon}$	State of strain at any point;
$\boldsymbol{\varepsilon}^{ep}$	Elastoplastic strain;
$\dot{\varepsilon}_{ij}^c$	Creep strain components;
$\boldsymbol{\varepsilon}^c$	Creep strain;
ε_i^v	Eigenvalues of the short-term creep strain tensor;
ε_{ij}^v	Short-term creep strain components;
$\boldsymbol{\varepsilon}^v$	Short-term creep strain;
ε_{ij}^{v+}	Components of positive part of short-term creep strain;
$\boldsymbol{\varepsilon}^{v+}$	Positive part of short-term creep strain;
ε_{ij}^f	Long-term creep strain components;
$\dot{\varepsilon}_{ij}^f$	Rate of long-term creep strain components;
$\boldsymbol{\varepsilon}^f$	Long-term creep strain;
$\boldsymbol{\varepsilon}^{ve}$	Viscoelastic creep strain;
$\boldsymbol{\varepsilon}^{vp}$	Viscoplastic creep strain;
$\boldsymbol{\varepsilon}^{vd}$	Viscodamage creep strain;
$\boldsymbol{\varepsilon}^0$	Shrinkage and thermal strain;
$\hat{\eta}$	Aging viscosity function;
η_0	Effective viscosity of the solidified matter;
θ	Time variable for changes in mechanical properties;
λ	First Lamé elastic constant and First Lamé non-aging relaxation function;

$\hat{\lambda}$	First Lamé aging relaxation function;
$\hat{\lambda}_f$	First long-term creep Lamé aging function;
$\widetilde{\lambda}$	First Lamé symmetric aging relaxation function for energy;
$\dot{\widetilde{\lambda}}$	Rate of change in first Lamé symmetric aging relaxation function for energy;
Λ	Rate of dissipation energy;
μ	Second Lamé elastic constant and Second Lamé non-aging relaxation function;
$\hat{\mu}$	Second Lamé aging relaxation function;
$\hat{\mu}_f$	Second long-term creep Lamé aging function;
$\widetilde{\mu}$	Second Lamé symmetric aging relaxation function for energy;
$\dot{\widetilde{\mu}}$	Rate of change in second Lamé symmetric aging relaxation function for energy;
$\dot{\xi}$	Rate multiplier defined through the consistency condition;
ξ^D	Multiplier associated to damage evolution;
$\dot{\xi}^D$	Rate multiplier defined through the consistency associated to damage;
ρ	Mass density;
σ_{ij}	Stress tensor components;
$\boldsymbol{\sigma}$	State of stress at the point;
τ	Auxiliary time variable for loading time;
τ'	Auxiliary time variable for loading time;
Φ^D	Dissipation inequality for damage;
Φ^f	Dissipation inequality for viscous flow;
χ_I	Material constant and relaxation non-aging material functions;
$\hat{\chi}_I$	Aging relaxation material function;
$\widetilde{\chi}_I$	Symmetric aging relaxation material function for energy;
$\dot{\widetilde{\chi}}_I$	Rate of change of symmetric aging relaxation material function for energy;
ψ	Helmholtz free energy per unit mas of solidified layer of concrete;
$\rho\Psi$	Instantaneous Helmholtz free energy per unit volume;
$\rho\Psi^{v\varepsilon}$	Strain energy associated to short-term creep strain, damage, and solidification;
$\rho\Psi^f$	Strain energy associated to viscous flow strain and volume of solidification;
$\rho\Psi^D$	Energy associated with the damage strengthening;
Ψ	Instantaneous Helmholtz free energy per unit mass;
Ω	Dissipation potential;
Ω^D	Dissipation potential for damage.

References

1. Keerthana, K.; Chandra Kishen, J.M. An experimental and analytical study on fatigue damage in concrete under variable amplitude loading. *Int. J. Fatigue* **2018**, *111*, 278–288. [CrossRef]
2. Bažant, Z.P.; Yu, Q.; Li, G.H.; Klein, G.J.; Křístek, V. Excessive deflections of record-span prestressed box girder. *ACI Concr. Int.* **2010**, *32*, 44–52.
3. Bažant, Z.P.; Hubler, M.H.; Yu, Q. Pervasiveness of Excessive Segmental Bridge Deflections: Wake-Up Call for Creep. *ACI Struct. J.* **2011**, *108*, 766–774.
4. Bažant, Z.P.; Hubler, M.H. Theory of cyclic creep of concrete based on Paris law for fatigue growth of subcritical microcracks. *J. Mech. Phys. Solids* **2014**, *63*, 187–200. [CrossRef]
5. Suaris, W.; Shah, S.P. Rate-Sensitive Damage Theory for Brittle Solids. *J. Eng. Mech.* **1984**, *110*, 985–997. [CrossRef]
6. Mazars, J.; Hamon, F.; Grange, S. A New 3D Damage Model for Concrete under Monotonic, Cyclic and Dynamic Loading. *Mater. Struct.* **2015**, *48*, 3779–3793. [CrossRef]
7. Richard, B.; Ragueneau, F. Continuum Damage Mechanics Based Model for Quasi-Brittle Materials Subjected to Cyclic Loading: Formulation, Numerical and Applications. *Eng. Fract. Mech.* **2013**, *98*, 383–406. [CrossRef]
8. Ortiz, M. A Constitutive Theory for the Inelastic Behavior of Concrete. *Mech. Mater.* **1985**, *4*, 67–93. [CrossRef]
9. Mazars, J.; Berthaud, Y.; Ramtani, S. The Unilateral Behavior of Damaged Concrete. *Eng. Fract. Mech.* **1990**, *35*, 629–635. [CrossRef]
10. Frémond, M.; Nedjar, B. Damage in Concrete: The Unilateral Phenomenon. *Nucl. Eng. Des.* **1995**, *156*, 323–335. [CrossRef]
11. Suaris, W.; Ouyang, C.; Fernando, V. Damage Model for Cyclic Loading of Concrete. *J. Eng. Mech.* **1990**, *116*, 1020–1035. [CrossRef]
12. Li, Q.; Zhang, L.; Ansari, F. Damage Constitutive for High Strength Concrete in Triaxial Cyclic Compression. *Int. J. Solids Struct.* **2002**, *39*, 4013–4025. [CrossRef]
13. Yazdani, S.; Schreyer, H.L. Combined Plasticity and Damage Mechanics Model for Plain Concrete. *J. Eng. Mech.* **1990**, *116*, 1435–1450. [CrossRef]
14. Abu-Lebdeh, T.M.; Voyiadjis, G.Z. Plasticity-Damage Model for Concrete under Cyclic Multiaxial Loading. *J. Eng. Mech.* **1993**, *119*, 1465–1484. [CrossRef]

15. Grassl, P.; Jirásek, M. Damage-Plastic Model for Concrete Failure. *Int. J. Solids Struct.* **2006**, *43*, 7166–7196. [CrossRef]
16. Wu, J.Y.; Li, J.; Faria, R. An Energy Release Rate-based Plastic-Damage Model for Concrete. *Int. J. Solids Struct.* **2006**, *43*, 583–612. [CrossRef]
17. Voyiadjis, G.Z.; Abu Al-Rub, R.K. A Plasticity and Anisotropic Damage Model for Plain Concrete. *Int. J. Plast.* **2007**, *23*, 1874–1900.
18. Häussler-Combe, U.; Hartig, J. Formulation and Numerical Implementation of a Constitutive Law for Concrete with Strain-based Damage and Plasticity. *Int. J. Non-Linear Mech.* **2008**, *43*, 399–415. [CrossRef]
19. Nguyem, G.D.; Korsunsky, A.M. Development of an Approach to Constitutive Modelling of Concrete: Isotropic Damage Coupled with Plasticity. *Int. J. Solid Struct.* **2008**, *45*, 5483–5501. [CrossRef]
20. Voyiadjis, G.Z.; Taqieddin, Z.N.; Kattan, P.I. Anisotropic Damage-Plasticity Model for Concrete. *Int. J. Plast.* **2008**, *24*, 1946–1965. [CrossRef]
21. Matallah, M.; La Borderie, C. Inelasticity-Damage-Based Model for Numerical Modeling of Concrete Cracking. *Eng. Fract. Mech.* **2009**, *76*, 1087–1108. [CrossRef]
22. Abu Al-Rub, R.K.; Kim, S.M. Computational Applications of a Coupled Plasticity-Damage Constitutive Model for Simulating Plain Concrete Fracture. *Eng. Fract. Mech.* **2010**, *77*, 1577–1603. [CrossRef]
23. Grassl, P.; Xenos, D.; Nyström, U.; Rempling, R.; Gylltoft, K. CDPM2: A Damage-Plasticity Approach to Modelling the Failure of Concrete. *Int. J. Solids Struct.* **2013**, *50*, 3805–3816. [CrossRef]
24. Chaboche, J.L.; Lesne, P.M.; Maire, J.F. Continuum Damage Mechanics, Anisotropy and Damage Deactivation for Brittle Materials Like Concrete and Ceramic Composites. *Int. J. Damage Mech.* **1995**, *4*, 5–22. [CrossRef]
25. Murakami, S.; Kamiya, K. Constitutive and Damage Evolution Equations of Elastic-Brittle Materials Based on Irreversible Thermodynamics. *Int. J. Mech. Sci.* **1997**, *39*, 473–486. [CrossRef]
26. Mazars, J.; Pijaudier-Cabot, G. From Damage to Fracture Mechanics and Conversely: A Combined Approach. *Int. J. Solids Struct.* **1996**, *33*, 3327–3342. [CrossRef]
27. Sain, T.; Chandra Kishen, J.M. Energy-based Equivalence between Damage and Fracture in Concrete under Fatigue. *Eng. Fract. Mech.* **2007**, *74*, 2320–2333. [CrossRef]
28. Papa, E.; Taliercio, A. Anisotropic Damage Model for the Multiaxial Static and Fatigue Behavior of Plain Concrete. *Eng. Fract. Mech.* **1996**, *55*, 163–179. [CrossRef]
29. Alliche, A. Damage Model for Fatigue Loading of Concrete. *Int. J. Fatigue* **2004**, *26*, 915–921. [CrossRef]
30. Desmorat, R.; Ragueneau, F.; Pham, H. Continuum Damage Mechanics for Hysteresis and Fatigue of Quasi-Brittle Materials and Structures. *Int. J. Numer. Anal. Methods Geomech.* **2007**, *31*, 307–329. [CrossRef]
31. Mai, S.H.; Le-Corre, F.; Nedjar, B. A Continuum Damage Modeling of Quasi-Static Fatigue Strength of Plain Concrete. *Int. J. Fatigue* **2012**, *37*, 79–85. [CrossRef]
32. Sellier, A.; Bary, B. Coupled Damage Tensors and Weakest Link Theory for the Description of Crack Induced Anisotropy in Concrete. *Eng. Fract. Mech.* **2002**, *69*, 1925–1939. [CrossRef]
33. Cope, D.; Yazdani, S.; Ju, J.W. A Method to Generate Damage Functions for Quasi-Brittle Solids. *J. Appl. Mech.* **2005**, *72*, 553–557. [CrossRef]
34. Tao, X.; Phillips, D.V. A Simplified Isotropic Damage Model for Concrete under Bi-axial Stress States. *Cem. Concr. Compos.* **2005**, *27*, 716–726. [CrossRef]
35. Richard, B.; Ragueneau, F.; Cremona, C.; Adelaide, L. Isotropic Continuum Damage Mechanics for Concrete under Cyclic Loading: Stiffness Recovery, Inelastic Strains and Frictional Sliding. *Eng. Fract. Mech.* **2010**, *77*, 1203–1223. [CrossRef]
36. Simo, J.C.; Ju, J.W. Strain- and Stress-Based Continuum Damage Models I. Formulation. *Int. J. Solids Struct.* **1987**, *23*, 821–840. [CrossRef]
37. Mazars, J.; Pijaudier-Cabot, G. Continuum Damage Theory, Application to Concrete. *J. Eng. Mech.* **1989**, *115*, 345–365. [CrossRef]
38. de Borst, R. Fracture in Quasi-Brittle Materials: A Review of Continuum Damage-Based Approaches. *Eng. Fract. Mech.* **2002**, *69*, 95–112. [CrossRef]
39. Park, T.; Ahmed, B.; Voyiadjis, G.Z. A review of continuum damage and plasticity in concrete: Part I–Theoretical framework. *Int. J. Damage Mech.* **2021**, *31*, 901–954. [CrossRef]
40. Bažant, Z.P. Phenomenological Theories for Creep of Concrete Based on Rheological Models. *Acta Tech. CSAV* **1966**, *31*, 82–109.
41. Bažant, Z.P.; Wu, S.T. Rate-type Creep Law of Aging Concrete Based on Maxwell Chain. *Mater. Constr.* **1974**, *7*, 45–60. [CrossRef]
42. Bažant, Z.P. Viscoelasticity of Solidifying Porous Material-Concrete. *J. Eng. Mech. Div.* **1977**, *103*, 1049–1067. [CrossRef]
43. Bažant, Z.P.; Prasannan, S. Solidification Theory for Concrete Creep. I: Formulation. *J. Eng. Mech.* **1989**, *115*, 1691–1703. [CrossRef]
44. Bažant, Z.P.; Prasannan, S. Solidification Theory for Concrete Creep. II: Verification and Application. *J. Eng. Mech.* **1989**, *115*, 1704–1725. [CrossRef]
45. Bažant, Z.P.; Huet, C. Thermodynamic Functions for Ageing Viscoelasticity: Integral Form without Internal Variables. *Int. J. Solids Struct.* **1999**, *36*, 3993–4016. [CrossRef]
46. Jennings, H.M.; Thomas, J.; Vlahnic, I. Can Nano-Models Lead to Improved Concrete? Materials Science as the Intersection of Chemistry and Mechanics. In Proceedings of the CONCREEP-8, Ise, Japan, 30 September–2 October 2008; pp. 11–23.
47. Bažant, Z.P.; Baweja, S. Creep and Shrinkage Prediction Model for Analysis and Design of Concrete Structures: Model B3. In *Adam Neville Symposium: Creep and Shrinkage—Structural Design Effects, ACI SP–194*; American Concrete Institute: Farmington Hills, MI, USA, 2000; pp. 1–83.

48. Mazzotti, C.; Savoia, M. Non-linear Creep Damage Model for Concrete under Uniaxial Compression. *J. Eng. Mech.* **2003**, *129*, 1065–1075. [CrossRef]
49. Papa, E.; Taliercio, A. A Visco-Damage Model for Brittle Materials under Monotonic and Sustained Stress. *Int. J. Numer. Anal. Methods Geomech.* **2005**, *29*, 287–310. [CrossRef]
50. Liu, W.; Zhou, H.; Zhang, S.; Jiang, S. Constitutive model of concrete creep damage considering the deterioration of creep parameters. *Constr. Build. Mater.* **2021**, *308*, 125047. [CrossRef]
51. Xu, Y.; Xu, Z.D.; Guo, Y.Q.; Jia, H.; Huang, X.; Wen, Y. Mathematical modeling and test verification of viscoelastic materials considering microstructures and ambient temperature influence. *Mech. Adv. Mater. Struct.* **2022**, *29*, 7063–7074. [CrossRef]
52. He, Z.; Shi, F.; Lin, Z.; Zhang, C.; Zhou, Y.; Zhao, F. Experimental characterization on cyclic stability behavior of a high-damping viscoelastic damper. *Constr. Build. Mater.* **2023**, *371*, 130749. [CrossRef]
53. Teran-Torres, B. Non-Linear Creep-Relaxation Constitutive Model for Plain Concrete with Anisotropic Damage. Doctoral Thesis, Illinois Institute of Technology, Chicago, IL, USA, 2016.
54. Christensen, R.M. *Theory of Viscoelasticity*, 2nd ed.; Dover Publications: Mineola, NY, USA, 2003.
55. Murakami, S. *Continuum Damage Mechanics: A Continuum Mechanics Approach to the Analysis of Damage and Fracture*, 1st ed.; Springer: Berlin/Heidelberg, Germany, 2012.
56. Swoboda, G.; Zaki, M. Application of Damage Mechanics to Deep Tunnels. In Proceedings of the 4th European Conference on NUMGE98, Udine, Italy, 14–16 October 1998; pp. 389–399.
57. Nair, S. *Introduction to Continuum Mechanics*; Cambridge University Press: New York, NY, USA, 2009.
58. Abu Al-Rub, R.; Darabi, M. A Thermodynamic Framework for Constitutive Modeling of Time- and Rate-Dependent Materials. Part I: Theory. *Int. J. Plast.* **2012**, *34*, 61–92. [CrossRef]
59. Martyushev, L.M.; Seleznev, V.D. Maximum entropy production principle in physics, chemistry and biology. *Phys. Rep.* **2006**, *426*, 1–45. [CrossRef]
60. Lubliner, J. *Plasticity Theory*; Dover Publications: Mineola, NY, USA, 2008.
61. Efthimiou, C. Cauchy's Equations. In *Introduction to Functional Equations: Theory and Problem-Strategies for Mathematical Competitions and Beyond*; MSRI Mathematical Circles Library: Berkeley, CA, USA, 2010.
62. Wang, W.; Gong, J. New relaxation function and age-adjusted effective modulus expressions for creep analysis of concrete structures. *Eng. Struct.* **2019**, *188*, 1–10. [CrossRef]